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EULER'S POWER CALCULATIONS OF "NATURAL FORCES" TO RAISE WATERS WITH PISTON PUMPS

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Abstract

Introduction: In a 1754 publication, *Discussion plus particulière de diverses manières d'elever de l'eau par le moyen des pompes avec le plus grand avantage* (Very detailed explanation of the different methods of raising water through pumps with the greatest effectiveness), Leonhard Euler (1707–1783) made extensive use of the concept of mechanical power in estimates of the power needed to raise waters with piston pumps, by means of natural forces such as human and horse force, running waters, and windmills. **Purpose of the study:** We aimed to revisit this publication to show to the modern reader Euler's pioneering approach in providing rational calculations of the power of natural forces needed to drive different machines to raise waters with piston pumps. **Methods:** After a brief historical review on the use of natural forces to drive machines and the evolution of the concept of mechanical power, the method employed was the examination and an annotated reproduction of the main formulation using Euler's original notation and ways of scientific writing of the time. **Discussion:** We address the evolution of hydropower and wind power, particularly for the generation of electricity, and also show that despite of its much lower attractiveness, there have been some attempts in the use of human and animal power in developing countries, particularly in applications that do not require large and constant amounts of power inputs.

Keywords

Piston pumps to raise water, history of piston pumps, power of natural forces, history of hydraulic machines.

Introduction

The need of providing mechanical power to drive machines has been a pressing issue for humans since the earliest civilizations as it is still today. In classical antiquity, virtually all work was done by man-power or animal power. Water power was used for pumping and in ancient industrial processes but probably not much before the first century BC (Landels, 1978). The dates are uncertain, but it is said that as early as 1700 BC, Hammurabi used windmills for irrigation in the plains of Mesopotamia (Golding, 1976). There is evidence that wind power was used in Afghanistan around 700 AD and by the Chinese back in 1200 AD (Golding, 1976). Harnessing water power and using it to drive machinery were apparently not explored until the early part of the first century BC. According to the geographer Strabo, a water-mill was built in the Pontus (near the modern Niksar, N. central Turkey) (Landels, 1978). The conceptions of these machines were probably transmitted and perfected from generation to generation. Certainly, there were virtually no guidelines to estimate the required inputs (force, work, and power) to achieve the desired outputs (flow rate, head, pressure, and efficiency); in other words, the technological knowledge as is known today was non-existent, and, perhaps, not even thought of.

It is argued (Landels, 1978) that the Mediterranean world, and particularly the Roman Empire, was dominated by the Greek culture, which might have had important effects on the scientific thought. This was characterized by a liking for stability, rest and permanence, and an opposing dislike of change and movement, which caused people to set a high value on the permanent and stable. As a result, their understanding of static conditions (e.g., hydrostatics) and mechanical problems not involving movement was acute, whereas their ideas on dynamics (e.g., ballistics) were incomplete and inaccurate. They spoke of velocities but hardly even began to study acceleration, which impaired the notion of inertia and kinetic energy.

The study of motion was also impaired by the lack of devices to measure short intervals of time, of the order of seconds. Moreover, philosophers of the time, such as Plato, and their followers adopted an anti-physical attitude, exalting the "pure" and theoretical sciences (such as geometry and astronomy), looking down on any research that was mechanical, or which had practical applications (Landels, 1978).

First Notions of Force, Work, and Power

It is possible to say that this state of affairs began to change with Galileo Galilei (1564–1642) and a mechanical treatise of his authorship, referred to by the author himself as *Mechanics (Le mecaniche*), which consisted of handwritten texts, copies of which obtained wide circulation in Europe in the first half of the 17th century, and eventually was published in 1634 in French by Marin Mersenne (1588–1648) under the title *Les Mécaniques de Galilée*, a year after the conviction of Galileo by the Holy Inquisition. Along with Galileo, he referred to mechanics in the plural to designate the part concerned with machines, as distinct from the denomination in the singular — mechanic — understood as the general theory for the conditions of rest (equilibrium) and natural motion of physical bodies.

According to Mariconda (2008), the original manuscript by Galileo was considered lost. Nonetheless, two copies appeared, one in a short version and another in a long version, the former written in 1593/1594, and the latter written in 1601/1602. Based on secondary sources that examined these publications, Mariconda showed that the basic explanation scheme employed by Galileo consists in showing that, as for the machines known at Galileo's time (lever, pulley, wedge, screw, inclined plane, and capstan), all can be reduced to a system of simple levers. The lever principle is extended from the static to the dynamic case, and the effect of velocity on the motive power is considered, where the motive power is the product of the weight (force) of a body and its velocity, taken by Galileo as a measure of the power being used.

Galileo's approach differed from the works of his predecessors in his strategy of introducing idealized conditions with the basic objective of eliminating friction, in order to think of an ideal machine. Galileo's predecessors knew that friction reduces performance (yield) of a machine, but none of them were led to, or were able to, think about the following problem: what would happen with a perfect machine? Or, more simply, what would happen with a frictionless machine?

Thus, according to Galileo's conception of machines, they all have the function of transmitting and applying force or power as effectively as possible. In this conception, it is possible to develop a quantitative evaluation for the performance of machines in terms of the product of the driving force used and speed, which corresponds to an important step towards the quantification of the power of a machine and opens the way for the elaboration of concepts such as work and energy, which are fundamental for the development of modern engineering.

It is generally accepted that the concept of force was first formally introduced into physics with Newton's second law of motion. Similar to "quantity of matter", the product of density and volume, Newton (1643–1727) proposed "quantity of motion" as a measure of the same, arising from velocity and quantity of matter conjointly (mv).

Since the quantity of motion is a scalar, Newton's definition implicitly treated velocity v as scalar speed rather than vector velocity $\vec{v}$. Hence the quantity of motion is not precisely identical to the modern concept of momentum, which is a vector quantity given by $m\vec{v}$. Based on the scalar quantification of "motion", Newton had to acknowledge that, contrary to the teachings of Descartes, the "quantity of motion" is not conserved.

The concept of work (force times distance), and power (work per unit of time) may have been first introduced by Descartes (1596–1650) as indicated by Belidor[a], in his famous *Architecture Hydraulique* (de Belidor, 1819). In the first chapter (§ 85), the concept was discussed by Belidor, with the subtitle "Descartes Principle for Mechanics", and it is presented as follows (Fig. 1): "… a body only has force as long as it is in motion, and this force will be all the greater when it will have, at the same time, more mass and more speed, as a rectangle will have more surface when it has a larger base and a greater height. Or, since this surface is expressed by the product of these two dimensions, similar is the force of a body, which is also called its 'quantity of motion', and should be expressed by the product of its mass and its speed…"

In § 89 Belidor generalized the Descartes Principle considering power ($P = Fv$) rather than quantity of motion (mv), to read: "… in the state of equilibrium, the force and the weight will be as the reciprocal ratio of their speed; and, therefore, the quantity of motion of the force will be equal to that of the weight …" This translates to the equality of powers: $P_{in} = P_{out}$. In the subsequent paragraphs, de Belidor applied this principle to several simple mechanical machines such as: levers, pulleys, cranks, etc. Fig. 2 is an image of an ancient crane, composed of several mechanical elements for power conversion.

In § 99 Belidor exemplified the mechanical principle by applying it to a hoisting machine: "… a force of 25 livres[b] may aid a machine to hoist a weight of 500 livres, if the weight is only one foot of a way in

[a] Bernard Forest de Belidor (1697–1761), a military engineer, taught mathematics at the artillery school at La Fère where he authored several textbooks. Seeking to introduce mathematics into practical engineering, he wrote *La science des ingénieurs* (1729) and *Architecture hydraulique* (1737–1739).

[b] The livre poids de marc or livre de Paris was equivalent to about 489.5 grams and was used between the 1350s and the late 18th century.

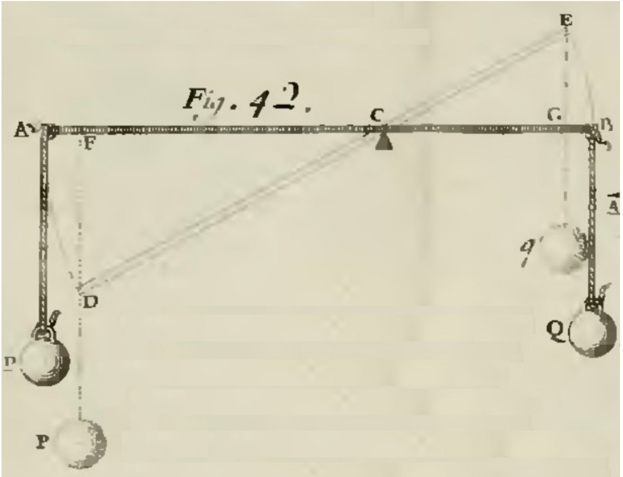

Fig. 1. de Belidor's illustration of the Descartes Principle for Mechanics (de Belidor, 1819)

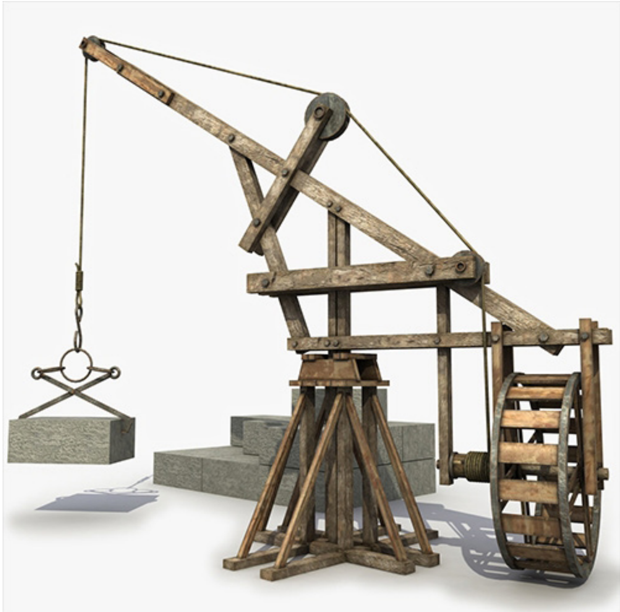

Fig. 2. Ancient crane hoisting a load with the aid of the human power applied to the paddles of the turning wheel (source: https://www.gruasyaparejos.com/en/construction-crane/ancient-crane/)

the time that the force will make 20 (feet): or else, a weight of 50 livres will raise a weight of 500 livres, if the former has a velocity ten times greater than that of the 500 livres weight …"

However, Claude-Louis Navier (1785–1836), the reviewer of the new edition of *Architecture Hydraulique*[c] , an academic at the *École Nationale des Ponts et Chaussées and École Polytechnique,* in a footnote following this paragraph, reproached de Belidor for his interpretation of the Descartes Principle and rephrased it in more rigorous terms by substituting speeds for virtual velocities, followed by other considerations. Nonetheless, de Belidor recast it in terms of power, perhaps because he was more interested in the practical applications of the principle.

Definition of power: if a constant force F is applied throughout a distance d, the work done T is defined as $T=F \cdot d$. In turn, the power P is defined as the work done per unit of time, $P=\frac{T}{t}=\frac{F \cdot d}{t}=F \cdot v$, where v is the speed. For a distance d equal to one meter, a force F equal to one newton, and for a unit of time of one second, the power is equal to one watt: $1W=1N \cdot 1\ m/s$.

Another common and traditional measure of power is the horsepower (hp)[d], comparing to the power of a horse (Fig. 3); one mechanical horsepower equals about 735.5 watts. This means that a horse is capable of raising a load of 75 kg (165 lbf) to a height of 1 m (3.28 ft) in 1 s. Since the normal gravity is equal to 9.80665 m/s^2, then a force of 735.5 N is necessary to raise a mass of 75 kg, which at a velocity of 1 m/s, would require a power of foot-pounds per second (FPS).

When considering human-powered equipment, a healthy human can produce about 1.2 hp (0.89 kW) "briefly" and sustain about 0.1 hp (0.075 kW) "indefinitely".

[c] In the 1810s, de Belidor's two works, *La science des ingénieurs* (1729) and *Architecture hydraulique* (1737–1739) were issued in revised and expanded editions by Navier, who had been recruited by the École des Ponts et Chaussées to edit the works of his great-uncle, the great French engineer Émiland Gauthey. By 1813 Navier completed this task and also issued a revised and expanded edition of Belidor's *La science des ingénieurs*. Navier's success as an editor of Belidor's *Science des ingénieurs* and Gauthey's works led their publisher, Firmin Didot, to invite him to prepare a revised edition of Belidor's *Architecture hydraulique*. Navier sought to correct the errors found in this work and give it a mathematical sophistication that would make it useful to the graduates of the École Polytechnique. Navier's contributions to the *Architecture hydraulique* are confined to the first volume, which contains notes and commentary equal or surpassing the original text in length. The remaining volumes consist of reissues, with new titles dated 1810, of the edition published in 1780 (text adopted from Jeremy Norman's Historyofscience.com, *https://www.jnorman.com/cgi-bin/hss/38462.html,* accessed on March 9, 2017).

[d] Until the mid-18th century, most demanding labor required horses. With the advent of the steam engine, machinery began to replace horses for various tasks. But many people resisted this change; they were skeptical about the efficiency of the new machines. Inventor James Watt (1736–1819) knew this and capitalized on it when marketing his improved steam engine. Watt noticed people's reticence to adopt the new technology and decided to make a measurement comparison that potential buyers could relate to: horses. But instead of figuring out exactly how much power a horse really produced, he estimated it. Watt guessed that a pony could lift an average of 220 lbf (pound-force) 100 ft. per minute (220 lbf x 100 ft./min. = 22,000 lbf x ft./min.). From there, he extrapolated that a horse could lift 50 percent more than a pony, bringing the estimated power of a horse, or horsepower, to 33,000 lbf x ft./min = 550 lbf x ft./s. Regardless of how accurate his measurements actually were (some neigh-sayers disagreed with them because no horse could sustain that level of effort for an extended period of time), the comparison was an effective one, and the term stuck (text adopted from "The History of Horsepower" by Paul Humphreys, https://www.thecompressedairblog.com/the-history-of-horsepower, accessed on March 10, 2021).

Euler's Calculations of Mechanical Power from "Natural Forces"

Leonhard Euler (1707–1783) was, perhaps, the first to formally introduce mechanical power as a measure of the capacity of natural forces to drive piston pumps by means of humans, horses, running waters, and windmills (Euler, 1754). This is a most remarkable publication from the engineering point of view, because Euler was capable of providing detailed calculations on how to design systems to extract power from these natural sources, which can well serve as an introductory historic chapter to any textbook on the evolution of machinery for raising waters.

Euler began by invoking his previous memoir on raising waters with piston pumps (Euler, 1754b), in which he had developed the analytical tools necessary for determining the pressures that the piping system should sustain according to the required raising height and volume flow rate. From these, and from the pump dimensions, he was able to determine the piston velocity and the effort exerted by the pump's piston per second (equal to the work done per second during the pump's piston motion, or the supplied power).

Fig. 4 shows a conception of a pair of piston pumps that operate out of phase in their delivery and aspiration cycles, in order to provide a continuous flow of water through the piping system. In this figure, F is the force applied to the lever that is supposed to move with a velocity a, and K is the resulting force applied to the piston that is supposed to move with a velocity ζ, such that $K\zeta = Fa$. The pumping system is supposed to have the following characteristics: piston diameter = a, piston excursion = b, diameter of the piping system = c, height of the reservoir = g, length of the pipe = l, cycling time of the piston = t, flow rate per hour = M, and pressure inside the pipe at the pump exit = p.

Next, Euler wrote the following formulas, which he had obtained in his previous memoir (Euler, 1754b). For the cycling time of the piston:

$$t = \frac{0.4484a^2\sqrt{bl}}{c\sqrt{\left(K - \frac{\pi a^2}{4}g\right)}}(s), \tag{1}$$

for the flow rate:

$$M = \frac{3600F \propto}{\lambda g}\left(\frac{ft^3}{h}\right), \tag{2}$$

and for the pressure at the pump exit:

$$p = \frac{4K}{\pi a^2}(ft), \tag{3}$$

where the force K is given in cubic ft of water.

Euler also defined two additional parameters: the ratio of pressures λ, and the ratio of velocities i, which are written as follows:

$$\lambda = \frac{p}{g}, \tag{4}$$

$$i = \frac{\zeta}{\propto}, \tag{5}$$

and from this latter expression, the following relations hold: $\zeta = \frac{2b}{t} = i \propto$ and $t = 2b / i \propto$.

Setting the number of all pumps to be used = $2n$, where n is the number of pairs of pumps, Euler then provided the following additional formulas:

$$\lambda = \frac{1}{2} + \sqrt{\left(\frac{1}{4} + \frac{0.0815F \propto^2 il}{bc^2g^2}\right)}, \tag{6}$$

for the piston diameter:

$$a = \sqrt{\frac{1.2732F}{\lambda ing}}, \tag{7}$$

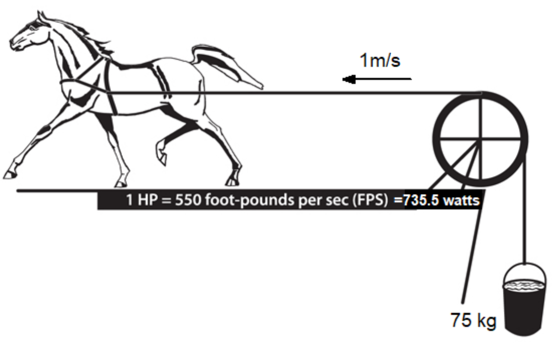

Fig. 3. One horsepower is needed to lift 75 kilograms (of water) by 1 meter in 1 second (adopted from https://aces.nmsu.edu/pubs/_m/M227/welcome.html)

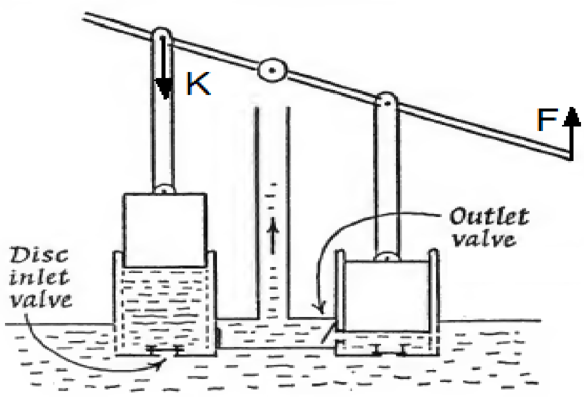

Fig. 4. A pair of piston pumps operating out of phase to provide a continuous flow of water through the piping system (Landels, 1978)

for the force acting on the piston:

$$K = \frac{1}{4}\lambda\pi a^2 g, \tag{8}$$

and for the piston velocity:

$$\zeta = \frac{1.2732\,F\alpha}{\lambda a^2 g}. \tag{9}$$

It will become clearer later that $1/\lambda$ is a measure of the system efficiency, which was extensively used by Euler for the optimization of the pumping system performance.

Man-power: by defining f as a force that a man is capable to deliver at rest and φ as the greatest velocity that a man can walk without too much fatigue, such that if a man has to walk at such velocity, he would be unable to exert any force, because all his efforts will be consumed through this course of walk. If ω is a velocity smaller than φ and ρ is the force that a man can proceed with the velocity ω, then, the relation between the forces f and ρ, can be obtained from the velocities φ and ω, by considering the following conditions:

1. If ω = 0, then $\boldsymbol{\rho} = f$,
2. If $\omega = \varphi$, , then $\boldsymbol{\rho} = 0$.

Euler then proposed the following ad-hoc analytical expression for $\boldsymbol{\rho}$:

$$\boldsymbol{p} = f\left(1-\frac{\omega}{\varphi}\right)^2. \tag{10}$$

To find the velocity that would correspond to the maximum deliverable power (which Euler called the greatest "moment of motion"), he proposed the differentiation with respect to ω of the following expression (which corresponds to the deliverable power):

$$f\omega\left(1-\frac{\omega}{\varphi}\right)^2, \tag{11}$$

under the supposition that the velocity ω is the variable that should be maximized, $\omega = \frac{1}{3}\varphi, \boldsymbol{p} = \frac{4}{9}f,$ and then the maximum deliverable power is $= \frac{4}{27}f\varphi$.

Considering that a man at rest can exert an effort of 60 lbf, and that without too much fatigue, he is able to follow a path at 6 ft per second, then f = 60 lbf and φ = 6 ft. Then, to apply most advantageously the force of a man to a given machine, it will be necessary for him to march at 2 ft per second, and the force will assume the following value: $\frac{4}{9}\cdot 60\,lb = 26\ 67\,lb$ [e]. Reducing this force to the weight of a volume of water, at a ratio of 70 lbf per cubic ft, this force will be equivalent to $\frac{8}{21}$ cubic ft (of water). Therefore, we can say that the force of a man is applied in the most advantageous way with a velocity of 2 ft per second, carrying a weight of $\frac{3}{8}ft^3$ of water.

If the number of men that one wishes to employ in a machine is set = m, and these men put the machine into motion with a velocity of 2 ft per second, their force ($= F$) will be $F = \frac{3m}{8}$ cubic ft of water, and the force that drives each pump will be $K = \frac{3m}{8in}$ cubic ft of water. For a velocity of 2 ft per second, $\propto = 2$, and the power of this force will be $F\propto = \frac{3}{4}m$. As a consequence, according to Eq. 2, the amount of water that will be raised in one hour will be $M = \frac{2700\,m}{\lambda g}$. Since a portion of that force will be used to overcome friction and raise the pistons to admit water by suction, the quantity M will be a little less, or it will be necessary to employ a few more men to overcome the obstacles.

Substituting $F\propto^2 = \frac{3}{2}m$ into Eq. 6, we obtain the following:

$$\lambda = \frac{1}{2} + \sqrt{\left(\frac{1}{4} + \frac{0.1222\,mil}{bc^2 g^2}\right)}. \tag{12}$$

Based on Eq. 7, we also obtain the diameter a of each piston:

$$a = \sqrt{\frac{0.4774\,m}{\lambda ing}}. \tag{13}$$

According to Eq. 4, the pressure that the pipe should sustain at its lower end ρ will be equivalent to the height λg.

Example of application: by putting m = 1 (one man driving pumps), $i = \frac{\zeta}{\propto} = \frac{6}{2} = 3$, n = 1, (one pair of pumps), and g = 30 ft in Eq. (13), we will obtain $\lambda = \frac{0.053}{a^2}$. By assuming b = 1 ft, $c = \frac{1}{12}ft = 1in$, and $l = 45$ ft, we will obtain $\lambda = 2.20$ and $a = 0.155$ ft. Based on Eq. 2, the flow rate will be $M = 41\ ft^3/h$, and based on Eq. 4, the pressure that the pipe should sustain at the pump exit will be $\rho = \lambda g = 66\ ft$. These results show that a man driving a pair of pumps, such as that shown in Fig. 4, is capable of rising 41 ft^3 of water per hour to a height of 30 ft through a pipe with a diameter of 1 in and 45 ft in length. The piston of each pump will have a diameter of 0.155 ft, each running into a cylinder of 1 ft in height. Of course, these results do not take into account the friction between the piston and the walls of the cylinders,

[e] These values would give approximately 0.1 hp (0.075 kW), which is the same value that has been proposed for the human power as mentioned earlier.

the friction of the water flowing through the pipe, and the power needed to raise the pistons to admit water by suction.

Horsepower: considering that for a horse $f=420lbt$ and $\varphi=12\ ft$, and based on the same power model established for a man (Eqs. 10 and 11), Euler found that the force of a horse is applied in the most advantageous way, with a velocity of 4 ft/s, carrying a weight $2\frac{2}{3}ft^3$ of of water (186.5 lbf).

Then, for a horse, the corresponding formula for λ (Eq. 6) is as follows:

$$\lambda=\frac{1}{2}+\sqrt{\left(\frac{1}{4}+\frac{3.4773mil}{bc^2g^2}\right)}, \qquad (14)$$

and, from Eq. 7, the diameter of each piston a will be as follows:

$$a=\sqrt{\frac{3.3952m}{\lambda ing}}. \qquad (15)$$

Running water power: the points $A, B, C, D, E, etc.$ along the circumference of the water wheel with the center at 0 are considered (Fig. 5), which is garnished with paddles Aa, Bb, Cc, etc receiving successive impulses of the running water lm: such that the water wheel by its motion drives the machine under consideration. Then, m is set as the center of the efforts of the water on the paddle Aa, which will fall roughly in its middle.

Let us set: the radius of the water wheel $0m=r$, the height of the paddle $Aa=h$, the length of the paddles, or the width of the water wheel $=f$, and the surface of each paddle $=fh$ velocity of the water wheel at point $m=v$ in ft per second, velocity of the running water $lm=e$ in ft per second.

Let us also consider that the paddle Aa is in the vertical position, and that the water hits only this paddle, such that the neighboring paddles Hh, Bb are above the free surface of the river.

Based on these definitions, the relative velocity of the water on the paddle will be equal to $(e-v)ft/s$, which is due to the height u. Considering that a weight falling from a height of 15.625 ft gives a velocity of 31.25 ft/s, then $\frac{(e-v)^2}{u}=\frac{31.25^2}{15.625}$, from which we have that $\frac{15.625(e-v)^2}{31.25^2}=\frac{(e-v)^2}{62.5}=\frac{2}{125}(e-v)^2$. The force of the water on this paddle will be equal to the weight of a volume of water $=\frac{2fh}{125}(e-v)^2$, from which the power ("moment of motion") is $=\frac{2fhv}{125}(e-v)^2$; and this will be as high as possible if $v=\frac{1}{3}e$. Therefore, to take this advantage, it will be necessary to arrange the machine in such a way that the wheel turns with such motion that the velocity at the center of the paddles is equal to one-third of the velocity of the running water; and then the force of the water that is applied to the wheel will be $=\frac{4}{9}\frac{2e^2fh}{125}=\frac{8}{1125}e^2fh$.

If our machine is set into motion by such a water wheel, we will have the force $F=\frac{8}{1125}e^2fh$ and the velocity of the wheel at a distance $0m=r$ will be $=\frac{1}{3}e$, which is the value of a, that is $a=\frac{1}{3}e$ and $F\alpha=\frac{8}{3375}e^3fh$. Therefore, the amount of water that could be raised in one hour by this machine is $M=\frac{8\cdot 3600}{3375\lambda g}e^3fh=\frac{123}{15}\frac{e^3fh}{g\lambda}$.

Then, for the water wheel, the corresponding formulas for λ and the diameter of the piston a are as follows:

$$\lambda=\frac{1}{2}+\sqrt{\left(\frac{1}{4}+\frac{ie^4fhl}{15529bc^2g^2}\right)}; \qquad (16)$$

$$a=\sqrt{\frac{2e^2fh}{221\lambda ing}}. \qquad (17)$$

According to Euler, a machine of this kind was put into practice at the Notre-Dame bridge in Paris (Fig. 6) to raise water to a height of 81 ft above the level of the Seine River. It was composed of two water wheels pushed by the flow of the river, each one driving a separate equipment.

Euler then commented that these wheels are similar to the one considered here, which according to a description by M. de Belidor[g] had the height h of the paddles equal to 3 ft and their width f equal

[g] de Belidor himself, in the *Avant-Propos* of the 1739 edition of *Architecture Hydraulique* (Vol. 2), reported on a call to the Notre-Dame facility: "… Messrs. The Provost of Merchants & Aldermen of the City of Paris having learned that I had commented on faults in the pumps of the machine applied to the bridge Notre-Dame, which supplies water from the Seine river to the greater number of public fountains, made me the honor of inviting me in 1737 to communicate to them my views on how to rectify this machine, in order to make it capable of a bigger product. As in working on the project that we carried out, it happened to me make several new discoveries on the movement of waters and the perfection of machines suitable for raising them, I have believed it necessary to suspend the printing of this Volume in order to infer them, and at the same time make essential corrections in several places, based on a few hydraulic principles, commonly received, the error of which I saw, as we can convince ourselves…"

to 18 *ft*. The velocity of the river e was estimated as 9 *ft/s*, and g=81 *ft*, which was equal to the length of the pipe l , because it was mounted vertically. Euler then argued that this machine would be capable of delivering the amount of water $M = \frac{4147}{\lambda} ft^3/h$, if "well managed", and twice this value considering both equipments, which increases as the value of λ is reduced.

However, according to Euler, the system proved to be capable to raise only 2400 ft^3/h, which de Belidor himself recognized that it was too little, and among other considerations, he attributed this result to the slow motion of the wheels, because just one-third of the paddles was immersed into the river, and not up to their centers. Corrections in the valves were also considered to reduce the flow resistance, and by these corrections, Belidor expected to more than double the output of the system.

The value λ of estimated by Euler for this system, considering both equipments, was $\lambda = \frac{8294}{2400} = 3\frac{1}{2}$, and, therefore, the pipe had to resist a pressure corresponding to more than three times the water column. Euler then concluded that the machine was highly defective in delivering much less water than it was desired, and as a consequence, the pressure inside the pipe increased considerably, which could cause the "destruction of the machine".

Here, it is possible to consider that $1/\lambda$ is a measure of the system efficiency; then $1/\lambda \times 100\,\% = 1/3\frac{1}{2} \times 100\,\% \approx 28\ 6\%$, which seems to be a reasonable estimate for the efficiency of a system of this sort.

Next, Euler proposed to improve the performance of the machine by reducing the value of λ as much as possible. His strategy was to increase the number of cycles for each complete turn of the wheel. He began by calculating the value of λ for $e = 9$, $f = 18$, $h = 3$ $l = g = 81$ and by considering that for each turn of an 8.5 *ft* radius wheel (with a peripheral velocity of 3 *ft/s*), each piston would accomplish μ cycles per turn of the wheel, giving then the following result:

$$\lambda = \frac{1}{2} + \sqrt{\left(\frac{1}{4} + \frac{\mu}{95c^2}\right)}\,. \tag{18}$$

By this approach, the value of λ for $c = \sqrt{2}$ would be reduced to 1.03 for μ = 6 and to 1.002 for μ = 4. According to the machine description, the piston excursion b=1.5 *ft*, which would result in a flow rate of 8052 ft^3/h for both wheels for μ = 6. Euler then considered that this output would be less, about 7200 ft^3/h, on account of the force required to overcome friction but, nonetheless, would triple the current output of the machine.

Euler finished this section by providing general design formulas for pumping water by means of

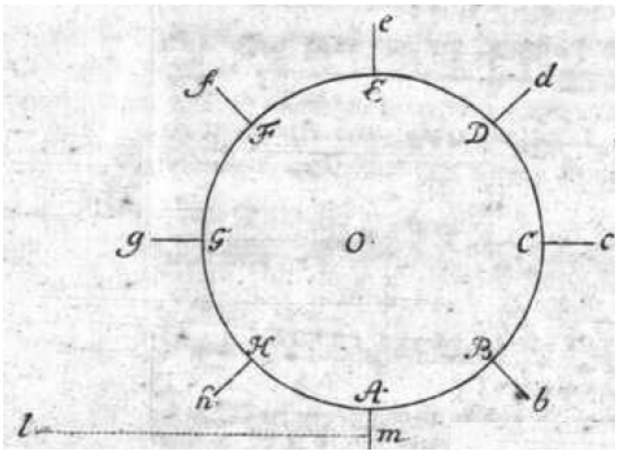

Fig. 5. Water wheel (Euler, 1754)

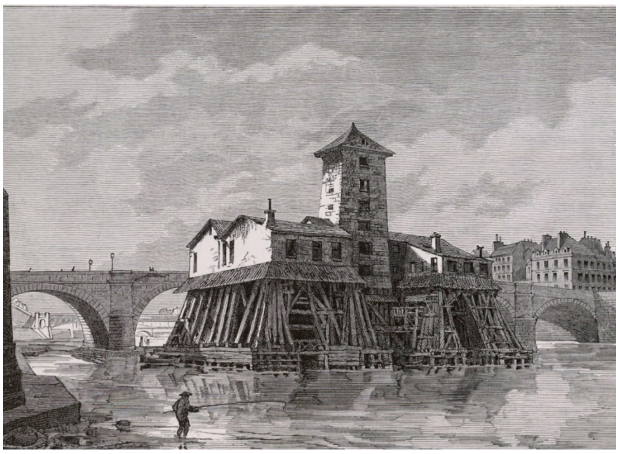

Fig. 6. Pompe Notre-Dame (Notre-Dame Pump) originally built in 1670 and reconstructed in 1708. This pump raised waters from the Seine into many public fountains and monuments, and it is said that it would allow the fountains to shoot water from 12 up to 50 feet in the air. Here shown in 1861, the pump looks rather destitute and neglected next to the Pont-Notre Dame. The Pont underwent much renovation during the 19th century, while the pump itself remained virtually untouched (source: https://fr.wikipedia.org/wiki/Fichier:Pompe_Notre-Dame,_vue_prise_de_la_vo%C3%BBte_du_quai_de_G%C3%AAvres_en_1861.jpg)

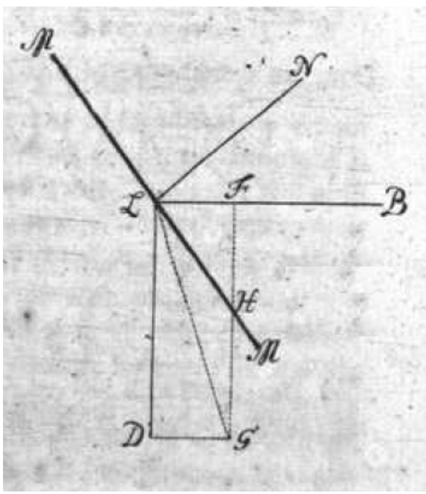

Fig. 7. Geometrical elements of the sail of a windmill (Euler, 1754)

water wheels, by substituting i in Eqs. 16 and 17 with $\frac{\mu b}{\pi r}$. This substitution is based on an ad-hoc assumption of the wheel turning with a peripheral velocity equal to one-third of the velocity of the river. However, Euler omitted the conditions on how to make the wheel to turn at this desired velocity. This would require a more complex and detailed calculation of the inertia of the system, including the mechanical characteristics of the water wheel itself, and other considerations such as friction (fluid and mechanical), which Euler was unable or had no desire to establish at that time.

Windmill power: Fig. 7 shows a view perpendicular to the keel of the sail of a windmill, where MLM is a section of the sail, the point L belongs to the keel, and the line LB represents the direction of the motion of point L. It is clear that the wind direction is in the same plane and that it is perpendicular to the line LB. The width of the sail $MM = y$, the inclination of the wind or angle $D\hat{L}N = \varphi$, the wind velocity $DL = e$, and the velocity of the point L of the sail $LF = v$; and since the sail "escapes" in part from the action of the wind, then one finds from the rules of mechanics that the effect would be the same if the wind is expressed by the diagonal GL, falling into the same direction GL when the sail is considered at rest.

The force of the wind on the line MM will be $=GL^2 \cdot sin^2\, GLM \cdot y$, which Euler showed that it can be written as $=y(e\, sin\varphi - v\, cos\varphi)^2$, and if the letters e and v are given in ft/s, this force will be equal to the weight of a volume of air, and then $=\frac{2}{125}y\left(e\sin\varphi - v\cos\varphi\right)^2$. But since air is about 800 times lighter that water, this force will be reduced to a mass of water whose volume $=\frac{1}{50000}y\left(e\sin\varphi - v\cos\varphi\right)^2$ in feet square, and assuming the width of the sail $MM = y$ is also expressed in feet, then, since we have considered it to be a line, the effect of one dimension out of three is still missing.

Since this force is in the direction LN, its component, which drives the line MM in the direction of the motion LB will be $=\frac{y\cos\varphi}{50000}\left(e\sin\varphi - v\cos\varphi\right)^2$, and if we call the infinitesimal width of this line dx, then the infinitesimal resultant force that makes the sail to turn in the direction LB will be $=\frac{ydx\cos\varphi}{50000}\left(e\sin\varphi - v\cos\varphi\right)^2 ft^3$ of water.

Fig. 8 is the frontal view of the sail $OAABB$ of a windmill, and Fig. 9 is an image of a real windmill, showing the elements of the sail considered in Fig. 7.

In these figures, $0D$ is the axis around which the sail turns, such that the sail falls into the wind direction. Here, $0CD = f$ is the sail span, and $OL = x$ is any section along the span, where the width $MLM=y$, and where the inclination of the wind direction over the element $MMmm$ is $=\varphi$. The velocity at the tip D of the sail will be set as u, and since $u:f=v:x$, then the velocity of point L will be $v=\frac{xu}{f}$; therefore, the wind force over the element $MMmm=ydx$ in the direction of the motion will be $=\frac{ydx\cos\varphi}{50000}\left(e\sin\varphi - \frac{xu}{f}\cos\varphi\right)^2 ft^3$ of water, where e is the wind velocity in ft/s. This element of force, when multiplied by the velocity $\frac{xu}{f}$, will give the element of the moment of motion for the force (element of power) of the wind over the sail, which will be $=\frac{xuydx\cos\varphi}{50000f}\left(e\sin\varphi - \frac{xu}{f}\cos\varphi\right)^2$.

Since, ordinarily, the angle φ is the same over the entire span of the sail, and assuming that the sail has the same width $AA=BB=h$, and because $y=h,$ the element of the moment of motion (element of power) will be given by

$$\frac{hu\cos\varphi}{50000f}xdx\left(e^2\sin^2\varphi - \frac{2eux}{f}sin\varphi\,\cos\varphi + \frac{u^2x^2}{f^2}\cos^2\varphi\right),$$

which after integration gives

$$\frac{hu\cos\varphi}{50000f}\left(\frac{1}{2}e^2x^2\sin^2\varphi - \frac{2eux^3}{3f}sin\varphi\,\cos\varphi + \frac{u^2x^4}{4f^2}\cos^2\varphi - C\right).$$

The constant C will be found by setting $x=OC$, for which the integral vanishes. Then, for $OC=k$,

$$C = \frac{1}{2}e^2k^2\sin^2\varphi - \frac{2euk^3}{3f}sin\varphi\,\cos\varphi + \frac{u^2k^4}{4f^2}\cos^2\varphi - C.$$

Let us set $x=OD=f,$ and the moment of motion (power) over the entire sail will be given by

$$\frac{hu\cos\varphi}{50000f}\left(\begin{array}{l}\frac{1}{2}e^2\left(f^2-k^2\right)\sin^2\varphi - \frac{2eu}{3f}\left(f^3-k^3\right)sin\varphi \\ \cos\varphi + \frac{u^2}{4f^2}\left(f^4-k^4\right)\cos^2\varphi\end{array}\right),$$

and if the windmill is equipped with four of such sails, the moment of motion for the force (power) of the wind will be given by the following:

$$F \propto = \frac{hu\cos\varphi}{12500f}\left(\begin{array}{l}\frac{1}{2}e^2\left(f^2-k^2\right)\sin^2\varphi - \\ \frac{2eu}{3f}\left(f^3-k^3\right)sin\varphi\,\cos\varphi + \\ \frac{u^2}{4f^2}\left(f^4-k^4\right)\cos^2\varphi\end{array}\right). \quad (19)$$

Upon differentiation of Eq. 19, Euler then found the maximum velocity that the sails should turn at their extremities D :

$$u = \frac{ef\tan\varphi}{9\left(f^4-k^4\right)}\left[\frac{8\left(f^3-k^3\right)\mp\left(f-k\right)}{\sqrt{\begin{array}{l}10f^4+20f^3+84f^2k^2+ \\ 20fk^3+10k^4\end{array}}}\right], \quad (20)$$

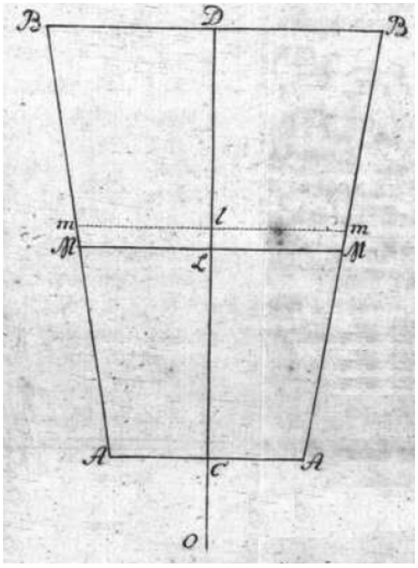

Fig. 8. Frontal view of the sail of a windmill (Euler, 1754)

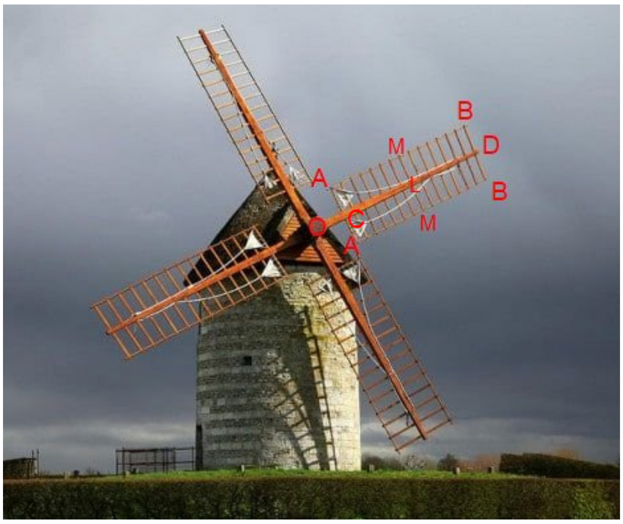

Fig. 9. Image of a real windmill, showing elements of the sail (without the sail cloth) considered in Fig. 8 (adapted from https://www.conserve-energy-future.com/how-windmills-work.php)

where the minus sign before the radical should be chosen for the maximum motion to be achieved.

Since the distance $OC=k$ can be chosen as small as possible, then it is allowed to consider $k=0$ to obtain $u=\frac{e\tan\varphi}{9}\left(8\mp\sqrt{10}\right)$. By setting $\frac{\left(8\mp\sqrt{10}\right)}{9}=\delta$, such that $u=\delta e\tan\varphi$, from Eqs. 19 and 20, the moment of motion for the total force (total power) of the wind will be given by the following:

$$F\propto=\frac{68\mp5\sqrt{10}}{729}\cdot\frac{e^3 fh\sin^3\varphi}{12500}. \tag{21}$$

Whereas the maximum motion will be obtained with the minus sign before the radical, giving a tip velocity $u=\delta e\tan\varphi=0.537525e\tan\varphi$, and as Eq. 21 shows, the maximum total power will be obtained, instead, with the plus sign before the radical, which is then given by the following:

$$F\propto=\frac{68+5\sqrt{10}}{729}\cdot\frac{e^3 fh\sin^3\varphi}{12500}=\frac{e^3 fh\sin^3\varphi}{108726}. \tag{22}$$

It is seen that the maximum action of the wind would be obtained by setting φ as a right angle; however, this would be unfeasible because the force F will be zero, since it varies with $\cos\varphi$, and $\propto$ will go to infinite, since u varies with $\tan\varphi$. Next, Euler provided design formulas for u and $F\propto$ for three values of φ, 45°, 55°, *and* 60°.

Finally, Euler considered the case where φ =54°, 45', $\tan\varphi=\sqrt{2}$, giving a tip velocity $u=0.76018$, resulting in one revolution of the sails every $8.2654\frac{f}{e}=t$ seconds. In the case where the windmill is driving piston pumps to raise water, and considering that each piston would accomplish μ cycles per turn of the sails $\frac{2b}{i\propto}=t=\frac{8.2654f}{\mu e}$, $i=\frac{0.24197\mu be}{\propto f}$ we obtain .

From these, it is possible to write the following:

$$\lambda=\frac{1}{2}+\sqrt{\left(\frac{1}{4}+\frac{0.01972\,\mu eF\propto l}{c^2g^2}\right)}; \tag{23}$$

and

$$a=\sqrt{\frac{5.2617f\cdot F\propto}{\lambda\mu nbeg}}. \tag{24}$$

Considering that from Eq. 22, $F\propto=\frac{e^3 fh}{199743}$, then, from Eqs. 23 and 24 we have the following:

$$\lambda=\frac{1}{2}+\sqrt{\left(\frac{1}{4}+\frac{\mu e^4 hl}{10128960c^2g^2}\right)}; \tag{25}$$

and

$$a=\frac{ef}{195}\sqrt{\frac{h}{\lambda\mu bg}}. \tag{26}$$

From these expressions, it is then possible to determine the piston diameter = a, the piston excursion = b, the diameter of the piping system = c, and the number of cycles per turn of the sails μ, such that the value of λ becomes as small as possible. Not only the pressure at the exit of the pump, which is = λg, will be the smallest, but also the quantity of water that will be raised, which is given by Eq. 2 as $M=\frac{3600F\propto}{\lambda g}\left(ft^3/h\right)=\frac{3600e^3 fh}{199743\lambda g}\left(ft^3/h\right)$, will be the largest. Or else:

$$M=\frac{e^3 fh}{55.5\lambda g}\left(\frac{ft^3}{h}\right), \tag{27}$$

bearing in mind that e is the velocity of wind, f is the span of the sails, and h is their width, and that the sails should turn one revolution in $8.2654\frac{f}{e}$ seconds.

The Windmill at Sanssouci: as an example of application of the above formulation for windmills driving piston pumps, let us consider a windmill that, supposedly, was used to raise waters at Sanssouci, which was examined in the paper by Eckert (2002): "… Pumps, driven by a windmill, should raise the water of the Havel River to an elevated reservoir. This proposal was executed; it involved the construction of a water reservoir on top of a hill 150 feet above the river level, with a windmill-driven water pump half-way between the river and the reservoir. The water had to be guided by a canal from the river to the site of the pump station, from where it would be pumped through pipes up the hill and into the reservoir. … Construction began in the summer of 1748. The canal from the Havel River to the pump station was finished by November. The windmill and the pumps were finished by the end of the year. The mechanism used to transmit the motion of the windmill to the pumps was described as clumsy, but it seemed to work. The pumps also were connected to a mechanism that could be set into motion by horses (Göpelwerk) if there were no wind…"

In Euler's publication (Euler, 1754b), there is no description on the characteristics of the pumps, neither on the windmill at Sanssouci, because, as a matter of fact, this particular system was never directly mentioned in this publication, although, as commented above, it was certainly motivated by Euler's involvement with this water park in 1749. However, as far as the characteristics of the pump are concerned, and for the sake of the present exercise, we can consider the pump characteristics given by Euler in an example of application at the end of Euler's publication (Euler, 1754b) itself: the piston pump diameter $a = \frac{4}{3} ft$, and the piston excursion $b = 4\ ft$. Then, let us assume that $n = 1$ (one pair of pumps), and that each pump completes four cycles per turn of the sails, then $\mu = 4$.

Some information about the piping system at Sanssouci was also reported by Eckert (2002) as follows: "… at the first system trial, by 1749, the wooden piping system burst at the lower end, … eventually, by 1753, these were replaced by lead tubes with an inner diameter $c = \frac{1}{3} ft$ …" In this same publication, there are indications that the length of the piping system was (approximately) $l = 8000\ ft$.

No information could be found on the characteristics of the windmills at Sanssouci; just a brief comment by Eckert (2002) that, by 1753: "… a second windmill was constructed at a different site to raise water to the reservoir independently of the first one, but it never seems to have worked properly…" Nonetheless, to put some reality into this exercise, it is possible to guess some information from the image of the windmill shown in Fig. 9: the span of the sails $f = 50\ ft$, the width of the sails $h = 12\ ft$. Wind speed $e = 25\ ft/s$ (assumed), we obtain one revolution in $8.2654\frac{f}{e} = 8.2654\frac{50}{25} = 16.53$ seconds, or 3.63 RPM.

Then, from Eq. 25 we have the following:

$$\lambda = \frac{1}{2} + \sqrt{\left(\frac{1}{4} + \frac{\mu e^4 h l}{10128960 c^2 g^2}\right)} =$$

$$\frac{1}{2} + \sqrt{\left(\frac{1}{4} + \frac{4 \cdot 25^4 \cdot 12 \cdot 8000}{10128960 \cdot \left(\frac{1}{3}\right)^2 150^2}\right)} = 2.98,$$

giving a system efficiency of $1/\lambda = 33.6\%$.

From Eq. 27, the flow rate is as follows:

$$M = \frac{e^3 fh}{55.5\lambda g} = \frac{25^3 \cdot 50 \cdot 12}{55.5 \cdot 2.98 \cdot 150} = 378\left(\frac{ft^3}{h}\right).$$

The value of λ = 2.98 shows that to raise the water to a height of *150 ft*, the pressure that is developed at the lower end of the pipeline (pump exit) $p = \lambda g = 447 ft$. However, a common assumption by the fontainiers and hydraulic engineers at the time was to consider that the necessary pressure developed by the pump would be equivalent to the height of the reservoir, which, as pointed out by Euler, was obviously incorrect. The consequence of not recognizing these high pressures was certainly the main cause of failure of the wooden pipelines at Sanssouci, as reported by Eckert (2002).

As far as the performance of the system is concerned, the following remarks by Eckert (2002) speak for themselves: "… By the spring of 1754, an abundance of snow and rain together with the water-raising machine (described as "miserably slow") produced some tangible results. On Good Friday 1754, with a half-filled reservoir, the King was given a demonstration. But that day it was windy and the main fountain rose to only about half the height that the King had expected [100 ft]– and after an hour, the reservoir was empty…" This was not surprising because the output of only 378 $(ft^3/h) = 10.7\ (m^3/h)$ as estimated above would not be sufficient for the grandeur of the water park wished by King Frederick[h].

[h] A steam-driven water pump was inaugurated in 1842 to supply water from the Havel River to the Sanssouci Palace, which in 2017 was nominated a "Civil Engineering Historical Landmark in Germany".

Using Natural Sources of Power Today

The natural sources of power considered here are still in much use nowadays (Gasch and Twele, 2016). For instance, the power of running waters, or hydropower as is known today, is widely employed in hydroelectricity, or hydroelectric power, where electricity is produced from hydropower. In 2015, hydropower generated 16.6% of the world's total electricity and 70% of all renewable electricity and was expected to increase by about 3.1% each year for the next 25 years. Hydropower is produced in 150 countries, with the Asia-Pacific region generating 33% of global hydropower in 2013. China is the largest hydroelectricity producer, with 920 TWh of production in 2013, representing 16.9% of domestic electricity use (Atkins, 2003) (Hydroelectricity (n.d.). In: Wikipedia. Accessed March 21, 2021, https://en.wikipedia.org/wiki/Hydroelectricity).

Different classes of hydraulic turbines for hydroelectric power plants can be selected according to the available head and flow rate, and due to the advances in the design of these machineries, they can operate with efficiencies up to 90%.

Wind power, or wind energy, is the use of wind to provide mechanical power through wind turbines to turn electric generators for electrical power. Wind power is a popular sustainable, renewable source of power that has a much smaller impact on the environment compared to burning fossil fuels. Wind farms consist of many individual wind turbines, which are connected to the electric power transmission network (Bennert and Werner, 1989). Onshore wind is an inexpensive source of electric power, competitive with or in many places cheaper than coal or gas plants. Onshore wind farms have a greater visual impact on the landscape than other power stations, as they need to be spread over more land and need to be built away from dense population (von König, 1976). Offshore wind is steadier and stronger than on land, and offshore farms have less visual impact, but construction and maintenance costs are significantly higher. Small onshore wind farms can feed some energy into the grid or provide power to isolated off-grid locations (Wind power (n.d.). In: Wikipedia. Accessed March 22, 2021, https://en.wikipedia.org/wiki/Wind_power).

A wind turbine is a machine that converts kinetic energy from the wind into electricity. The blades of a wind turbine turn between 13 and 20 revolutions per minute, depending on their technology, at a constant or variable velocity, where the velocity of the rotor varies in relation to the velocity of the wind in order to reach a greater efficiency. The wind turbine is automatically oriented to take maximum advantage of the kinetic energy of the wind, from the data registered by the vane and anemometer that are installed at the top. The nacelle turns around a crown located at the end of the tower. The wind makes the blades turn, which start to move with wind speeds of around 3.5 m/s and provide maximum power with a wind speed 11 m/s. With very strong winds (25 m/s), the blades are feathered and the wind turbine slows down in order to prevent excessive voltages. The rotor (unit of three blades set in the hub) turns a slow axis that is connected to a gear box that lifts the turning velocity from 13 to 1,500 revolutions per minute. The gearbox transfers its energy through a fast axis that is connected to the generator, which produces the electricity (accessed March 22, 2021, https://www.acciona.com/renewable-energy/wind-energy/wind-trurbines/).

Besides being unreliable and because of its irregular and much lower power output, man and animal power has not developed in the same pace as hydropower and wind power. Nonetheless, there are still some attempts in the use of these natural sources of power for driving machineries for different purposes (Fuller and Aye, 2012; Phaniraja and Panchasara, 2009).

Fig. 10 shows an image from a patent of invention aiming to provide an animal powered mechanical device for water desalination. The machine is described as follows (US Patent No. US7387728B2, by Pathak et al., date of publication: June 17, 2008): a pair of bulls, capable of exerting more than 100 kg of draft and walking at a rate of 50 meters/minute is coupled to a mechanical link with the help of a rope. The pair of bulls generates a pressure of 300 psi and a discharge of 20 liters/minute, when it completes one rotation of 8 meters diameter circular path.

Fig. 11 shows an image from a patent of invention of a system and method for producing electricity using the biological energy of the muscles of animals like horses. The machine is described as follows (US Patent No. US20050161289A1, by Gomez-Nacer, date of publication: July 28, 2005): "… A system and method for generating electricity by means of increasing the velocity of an animal on a mechanical device directly or indirectly attached to the hoof or other part of the limbs of the animal to use the force of its muscle contraction and the force produced

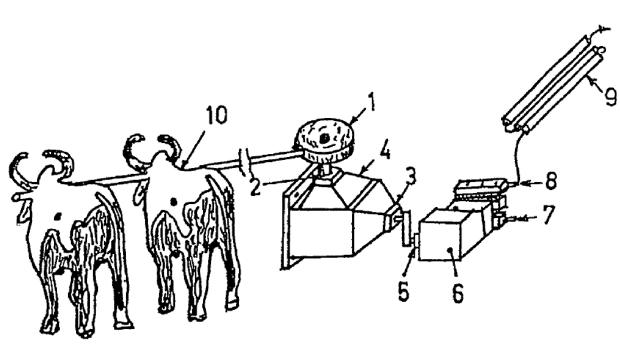

Fig. 10. Image from a patent of invention of an animal powered mechanical device for water desalination. Element # 9 is an osmosis membrane module (source: US Patent No. US7387728B2, by Pathak et al., date of publication: June 17, 2008)

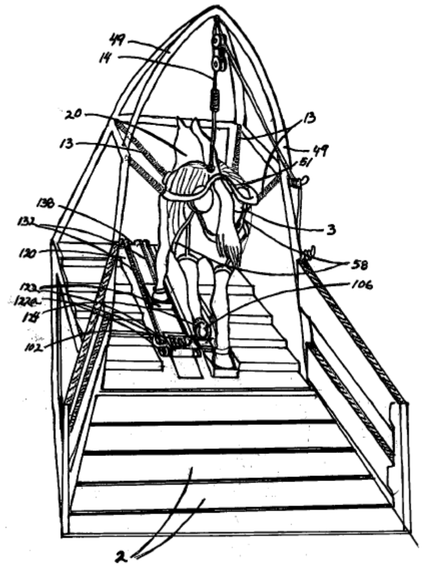

Fig. 11. Image from a patent of invention of a system and method for producing electricity using the biological energy of the muscles of animals like horses (source: US Patent No. US20050161289A1, by Gomez-Nacer, date of publication: July 28, 2005)

Fig. 12. Human muscle power operating a gear reduction system for charging automotive batteries (source: Yadav and Rao, 2015)

by its gravity to make spin multiplying wheels in communication to an electricity generator…"

Fig. 12 shows an image of a gear reduction system operated by human muscle power or animal power to drive a car alternator as a generator to charge a typical 12V 40Ah lead-acid automotive battery (not shown in the figure). In the experiments, human and animal power were used to charge the batteries. It took three hours to fully charge a 50% discharged battery and 1.5 hours to fully charge a 75% discharged battery.

Conclusions

Mechanical power as is known today, as a measure of the capacity to perform work, seems to have emerged in the first half of the 18^{th} century in the works of de Belidor, and was extensively used by Euler in his 1754 publication, in which he applied a pioneering approach in providing rational calculations for the power needed to drive different machinery to raise waters with piston pumps, by means of natural sources of power (human power, animal power, water flow power, and wind power). With the advent of the steam engine in the mid-18^{th} century, the horsepower as a measure of mechanical power was proposed, which later found its equivalent in the internationally standardized watt unit. It was shown that the use of hydropower and wind power have evolved considerably, particularly for the generation of electricity, and, despite of its much lower attractiveness, there have been some attempts in the use of human and animal power in particular applications that do not require large and constant amounts of power inputs.

References

Atkins, W. (2003). Hydroelectric power. *Water: Science and Issues*, Vol. 2, pp. 187–191.

Bennert, W. and Werner, U. J. (1989). *Windenergie*. Berlin: VEB Verlag Technik, 144 p.

de Belidor, B. F. (1819). *Architecture Hydraulique, ou l'art de conduire, d'élever et de ménager les eaux pour différents besoins de la vie. Nouvelle édition, avec des notes et additions par M. Navier, ingénieur du Corps Royal des Ponts -et-Chaussées. Première partie - Tome premier.* Paris: Firmin Didot, 666 p.

Eckert, M. (2002). Euler and the fountains of Sanssouci. *Archive for History of Exact Sciences*, Vol. 56, pp. 451–468. DOI: 10.1007/s004070200054.

Euler, L. (1754a). E206. Sur le mouvement de l'eau par des tuyaux de conduite. Mémoires de l'académie des sciences de Berlin, Vol. 8, pp. 111–148.

Euler, L. (1754b). E207. Discussion plus particulière de diverses manières d'elever de l'eau par le moyen des pompes avec le plus grand avantage. *Mémoires de l'académie des sciences de Berlin*, Vol. 8, pp. 149–184.

Fuller, R. J. and Aye, L. (2012). Human and animal power – The forgotten renewables. *Renewable Energy*, Vol 48, pp. 326–332. DOI: 10.1016/j.renene.2012.04.054.

Gasch, R. and Twele, J. (eds.) (2016) *Windkraftanlagen. Grundlagen, Entwurf, Planung und Betrieb*. 9th edition. Wiesbaden: Springer, 622 p.

Golding, E. W. (1976). *The generation of electricity by wind power*. London: E. & F. N. Spon Ltd., 332 p.

Kahlow, A. (2017). Das Pumpwerk für die Fontänen von Sanssouci. In: *Historische Wahrzeichen der Ingenieurbaukunst, Band 21*. Berlin: Bundesingenieurkammer, 92 p.

Landels, J. G. (1978). *Engineering in the ancient world*. Berkley & Los Angeles: University of California Press, 224 p.

Mariconda, P. R. (2008). As mecânicas de Galileu: as máquinas simples e a perspectiva técnica moderna. *Scientiae Studia*, Vol. 6, No. 4, pp. 565–606. DOI: 10.1590/S1678-31662008000400006.

Phaniraja K.L. and Panchasara H. H. (2009). Indian draught animals power. *Veterinary World*, Vol. 2 (10), pp. 404–407.

von König, F. (1976). *Windenergie in praktischer Nutzung*. Munchen: Udo Pfriemer Verlag, 184 p.

Yadav, D. K. and Rao, P. (2015). Study and analysis of animal and human muscle power for electricity generation. *International Journal of Engineering Research & Technology*, Vol. 3, Issue 20, IJERTCONV3IS20107.