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EVALUATING THE APPLICABILITY OF BERNOULLI'S HYPOTHESIS IN BEAM ANALYSIS

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Abstract

Introduction: In this paper, based on the properties of unit functions, we present accurate solutions to beam bending under various transverse loads and edge restraint conditions, using equations based on Bernoulli's hypothesis and the hypothesis taking into account transverse shears. By comparing the analytical solutions obtained for a rectangular beam, we determined beam length-to height (L/h) ratios for cases when the difference in deflections is less than the permitted value. Thus, criteria for Bernoulli's hypothesis application were obtained. The results of beam bending analysis can be applied when studying rod systems using the force and displacement methods. In this case, Bernoulli's hypothesis is used. All the ratios obtained are simple and clear. However, this hypothesis is applicable for the analysis of thin-walled structures. Meanwhile, the hypothesis taking into account transverse shears can be used for structures of medium cross-section height. To ensure accurate results when studying building structures (beams, plates, shells, rod systems), the criterion of Bernoulli's hypothesis (hypothesis of the straight normal) applicability was needed. **Purpose of the study**: We aimed to build a mathematical deformation model and develop a method for the analysis of bending in elastic Timoshenko beams with account for transverse shears. **Methods:** By applying generalized functions and direct integration of the differential equation for the bending line, we obtained analytical expressions for the deflection function under various boundary conditions. **Results:** Based on the proposed method, we performed beam analysis under various transverse loads and edge restraint conditions. We also evaluated the scope of Bernoulli's hypothesis application for the main types of beams used in the analysis of rod systems by the displacement method.

Keywords

Beam, bending, Kirchhoff model, transverse shear, Timoshenko model, unit functions.

Introduction

Latest advances in construction science show that a balanced combination of materials in a structure makes it possible to utilize their benefits to the maximum extent. Due to the widespread use of new structural materials ensuring structural efficiency, it is required to apply analytical models that would fairly represent the stress-strain state of structural elements in buildings and structures (Zveryayev, 2003). Many researchers explored how to build one-dimensional and two-dimensional approximate analytical models based on three-dimensional equations of elasticity theory (Donnell, 1982; Goldenweiser, 1976; Maslennikov, 2009; Nazarov, 2002; Tovstik, 2007; Zveryayev and Makarov, 2008).

To ensure a balanced combination of material properties, we need to make sure that such a combination is appropriate and provides the required load-bearing capacity of the structure while reducing its weight and manufacturing complexity, optimizing the construction period and operating expenses, thereby improving the performance of investment in construction, and justify that with analysis and calculations.

The widespread use of modern software systems for structural analysis in construction necessitates their verification to determine the reasonable level of detail with regard to the analyzed analytical model and the required accuracy of calculations (ANSYS, 2009; Simulia, 2012: SOFiSTiK AG, 2014). Thus, it is required to obtain accurate solutions for typical problems related to the analysis of new building structure types in order to use analytical solutions (Karpov et al., 2021) for verification of various software systems.

S. P. Timoshenko (Timoshenko, 1945; Timoshenko and Woinowsky-Krieger, 1963) is rightfully considered the author of the refined theory considering transverse shear in the analysis of beams, plates, and shells. He proposed an analytical model that takes into account bending and shear deformations and, thus, can be used to describe the behavior of beams of medium thickness as well as sandwich panels, and the high-frequency vibrations of beams when the wavelength becomes comparable to the cross-section height. In this case, the shear coefficient depends on Poisson's ratio. Numerous researchers attempted to obtain an exact expression for it (Cowper, 1966; Hutchinson, 1981; Stephen, 1980).

In engineering practice, the Timoshenko model (Timoshenko and Gere, 1976) is sufficient in most analysis cases. Based on the results of experimental studies conducted later, it was shown that, in the analysis of many building structures, the shear coefficient is underestimated (Franco-Villafañe and Méndez-Sánchez, 2016; Méndez-Sáchez et al., 2005).

Yeliseyeva et al. (2011) studied the application of the Timoshenko model in beam deflection analysis with account for bending and shear deformations. They showed that the resolving equation in the problem of accounting for additional shear in beam bending has terms with different physical meaning, which introduces particular aspects when boundary conditions are considered.

Lalin and Beliaev (2015) solved the problem of bending of a geometrically nonlinear cantilever beam, using the Kirchhoff and Cosserat–Timoshenko theories followed by a comparison of the results obtained. In their opinion, the findings can be used for verification of various software systems.

When classic beam bending problems are considered, Bernoulli's hypothesis is mainly applied. However, this hypothesis is not valid for, e.g., composite beams. The degree of approximation is mainly determined by the ratio between the cross-section height and the length of the beam as well as physical characteristics and structure of the material (Pavlenko and Vereshchaka, 2002). Rossikhin and Shitikova (2010) provided an analytical review of Timoshenko-type theories in respect to thin-walled open-section beams and concluded that currently there are no theories that would describe the beams under consideration and fully meet the requirements of engineering practice (analysis) and experimental data.

By using Bernoulli's hypothesis, Karpov et al. (2021) presented a method to find an accurate solution to the beam bending equation for a beam of uniform cross-section height, subjected to different types of transverse load (distributed along the entire length of the beam, distributed along a part of the beam, concentrated force, or a moment of a couple of forces), with different types of beam end restraint. An analytical solution for a beam can also be obtained by using the hypothesis taking into account transverse shears. By comparing these solutions, it is possible to determine criteria for Bernoulli's hypothesis application in beam analysis.

The equation for the equilibrium of a beam with length *L* and cross-section height *h*, subjected to the load *q*, when Bernoulli's hypothesis is used, is as follows:

$$EJw^{IV} = q, \tag{1}$$

where $J = h^3/12$ — the moment of section inertia, $w(x)$ — the beam deflection, $q(x)$ — the load (MPa).

If we apply the hypothesis taking into account transverse shears (Timoshenko model), then the equations for the equilibrium of such a beam will be as follows:

$$\frac{dQ_x}{dx} + q = 0, \ \frac{dM_x}{dx} - Q_x = 0, \tag{2}$$

where $Q_x = Gh\left(\psi_x + \frac{dw}{dx}\right), \ M_x = EJ\frac{d\psi_x}{dx}$.

Here ψ_x — the function taking into account transverse shears.

The method of solving Eq. (1) described by Karpov et al. (2021) can also be applied to solve system (2).

Direct integration of the differential equation for the bending line

Let us find a general solution of system (2) by direct integration of the differential equation for the bending line under different types of loads and boundary conditions. We will consider a case when the load *q* is uniformly distributed along the entire length of the beam. Let the beam be rigidly fixed at *x* = 0 and unrestrained at *x* = *L*. In this case, the following conditions must be fulfilled:

at *x* = 0, *w* = 0, and $\psi_x = 0$;

(3)

at *x* = *L*, $M_x = 0$ $\left(\frac{d\psi_x}{dx} = 0\right)$ and $Q_x = 0\left(\frac{d^2\psi_x}{dx^2} = 0\right)$.

Based on the second equation of system (2), we obtain the following:

$$Q_x = \frac{dM_x}{dx} = EJ\frac{d^2\psi_x}{dx^2}, \ \frac{dQ_x}{dx} = EJ\frac{d^3\psi_x}{dx^3}.$$

By substituting the obtained expressions into the first equation of system (2), we obtain the following:

$$EJ\frac{d^3\psi_x}{dx^3} = -q \quad \text{or} \quad \frac{d^3\psi_x}{dx^3} = -\frac{q}{EJ}. \tag{4}$$

Differential equation (4) represents an equation with separable variables. By integrating this differential equation successively, we get the following:

$$\frac{d^2\psi_x}{dx^2} = -\frac{q}{EJ}x + C_1,\ \frac{d\psi_x}{dx} = -\frac{q}{EJ}\frac{x^2}{2} + C_1x + C_2,$$
$$\psi_x = -\frac{q}{EJ}\frac{x^3}{6} + C_1\frac{x^2}{2} + C_2x + C_3. \tag{5}$$

By using boundary conditions (3), we find the following arbitrary constants:

$$C_3 = 0,\ C_1 = \frac{qL}{EJ}, C_2 = -\frac{qL^2}{2EJ}.$$

To find *w(x)*, let us use the following expression:

$$Q_x = Gh\left(\psi_x + \frac{dw}{dx}\right) = \frac{dM_x}{dx} = EJ\frac{d^2\psi_x}{dx^2};$$

i.e.,

$$Gh\left(\psi_x + \frac{dw}{dx}\right) = EJ\left(-\frac{q}{EJ}x + \frac{qL}{EJ}\right).$$

Hence, given that *G* = *E*/3, we obtain the following:

$$\frac{dw}{dx} = \frac{Eh^2 3}{12E}\frac{d^2\psi_x}{dx^2} - \psi_x.$$

By integrating this relation, we get the following:

$$w(x) = \frac{q}{EJ}\left(\frac{x^4}{24} - \frac{L}{6}x^3 + \frac{L^2}{4}x^2\right) + \frac{q}{EJ}\frac{h^2}{4}\frac{(L-x)^2}{2} + b_1. \tag{6}$$

By using the boundary conditions at *x* = 0, *w* = 0, we find b_1:

$$b_1 = -\frac{q}{EJ}\frac{L^2h^2}{8}.$$

Thus, the function *w(x)* will take the following form:

$$w(x) = \frac{q}{EJ}\left(\frac{x^4}{24} - \frac{L}{6}x^3 + \frac{L^2}{4}x^2\right) + \frac{q}{EJ}\left[\frac{h^2}{4}\frac{(L-x)^2}{2} - \frac{L^2h^2}{8}\right]. \tag{7}$$

If transverse shears are not considered (Kirchhoff model), then *w(x)* will be as follows:

$$w(x) = \frac{q}{EJ}\left(\frac{x^4}{24} - \frac{L}{6}x^3 + \frac{L^2}{4}x^2\right).$$

Therefore, since in this case transverse shears are considered, deflection (7) changed by Δ:

$$\Delta = \frac{q}{EJ}\left[\frac{h^2}{4}\frac{(L-x)^2}{2} - \frac{L^2h^2}{8}\right].$$

The maximum deflection will be at *x* = *L,* i.e.:

$$\Delta_{max} = -\frac{q}{EJ}\frac{L^2h^2}{8}.$$

To apply Bernoulli's hypothesis in this case, Δ_{max} must be small (NMT 5% of the permitted deflection w_{perm}). Based on this condition, we can find an estimate for the L/h ratio. For instance, for a rectangular concrete beam (E = 3•10[4]MPa, $w_{perm} = 0.0057h$, at q = 2•10[-2]MPa, we have $\frac{q}{EJ}\frac{L^2h^2}{8} = 0.05 \cdot 0.0057h$, hence, we can find $L=16h$. Thus, if $h>L/16$, then we need to use the model taking into account transverse shears. Based on the condition $w_{max} \le w_{perm}$, we obtain $L=30h$.

In the example considered, at $L=10m$, the permitted height of the beam *h* (when Bernoulli's hypothesis is applied) shall not exceed 0.625 m, and based on the condition $w_{max} \le w_{perm}$, the beam height turned out to be 0.33 m. If we need to increase the beam height based on the condition of structural integrity, then it can be increased by 0.295 m. In this case, the hypothesis of the straight normal remains valid.

For a rectangular metal beam (*E* = 2.1•10[5]MPa, $w_{perm} = 0.01h$, at q = 2•10[-2]MPa, L = 60*h.* At L = 10 m, $h = \le 0.16$ m, so that the beam bending equation with the use of Bernoulli's hypothesis could be applied in beam analysis.

Let us assume that the ends of the beam subjected to a load that is uniformly distributed along the entire length of the beam have a hinged support. In this case, at x = 0 and x = *L*, the following conditions must be fulfilled:

$$w=0, \frac{d\psi_x}{dx} = 0. \tag{8}$$

By using these boundary conditions, we will find arbitrary constants (except for C_3) and obtain the following:

$$\psi_x(x) = -\frac{q}{EJ}\frac{x^3}{6} + \frac{qL}{2EJ}\frac{x^2}{2} + C_3.$$

Based on $Gh\left(\psi_x + \frac{\partial w}{\partial x}\right) = EJ\frac{d^2\psi_x}{dx^2}$,

we will find *w(x)*. In this case:

$$\frac{dw}{dx} = \frac{q}{EJ}\left[\frac{x^3}{6} - \frac{L}{2}\frac{x^2}{2} - \frac{EJ}{q}C_3 - \frac{h^2}{4}\left(x - \frac{L}{2}\right)\right].$$

By integrating this relation, we get the following:

$$w(x)=\frac{q}{EJ}\left[\begin{array}{c}\frac{x^4}{24}-\frac{L}{12}x^3-\frac{EJ}{q}C_3x-\\ \frac{h^2}{4}\frac{\left(x-L/2\right)^2}{2}\end{array}\right]+b_1.$$

By using boundary conditions (8), we will find C_3 and b_1. Thus, $w(x)$ will take the following form:

$$w(x)=\frac{q}{EJ}\left\{\begin{array}{c}\frac{x^4}{24}-\frac{L}{12}x^3+\left(-\frac{L^3}{24}+\frac{L^2h}{32}\right)x-\\ \frac{h^2}{4}\left[\frac{\left(x-L/2\right)^2}{2}-\frac{L^2}{8}\right]\end{array}\right\}.$$

If transverse shears are not considered, then the deflection function will be as follows:

$$w(x)=\frac{q}{EJ}\left(\frac{x^4}{24}-\frac{L}{12}x^3-\frac{L^3}{24}x\right).$$

Therefore, since in this case transverse shears are considered, the deflection changed by Δ:

$$\Delta=\frac{q}{EJ}\frac{h^2}{4}\left[\frac{L}{8}x-\frac{\left(x-L/2\right)^2}{2}+\frac{L^2}{8}\right].$$

Since the maximum deflection will be at $x=L/2$, then $\frac{9qL^2}{16Eh}$ must be small. In case of a concrete beam, we have the following ratio: .

$$\frac{9qL^2}{16Eh}=0.05\cdot 0.0057h.$$

Therefore, to apply Bernoulli's hypothesis in beam analysis, the following condition must be fulfilled: *h<L/27*. If *h>L/27* , then we need to use the model taking into account transverse shears. For a concrete beam, at *L* = 10 m, the cross-section height *h* shall not exceed 0.37 m.

For a metal beam, L/h = 95, therefore, if $h>L/95$, then we need to use the model taking into account transverse shears. For instance, at L = 10 m, the beam cross-section height h shall not exceed 0.105 m.

In the same way, we can analyze Bernoulli's hypothesis applicability for beam analysis in case of other types of loads and beam end restraint.

Let the load be uniformly distributed along a part of the beam span, i.e.:

$$q(x)=q_1\bar{\delta}(x-x_1)=q_1\left[u(x-a_1)-u(x-a_2)\right],$$

where $u(x-a_1)$ and $u(x-a_2)$ are unit functions.

If the beam is rigidly fixed at $x=0$ and unrestrained at $x=L$, then the boundary conditions will take the form corresponding to that in system (3). In this case, Eq. (4) will take the following form:

$$\frac{d^3\psi_x}{dx^3}=-\frac{q_1}{EJ}\left[u(x-a_1)-u(x-a_2)\right]. \qquad (9)$$

By using the properties of unit functions

$$\int u(x-a_1)dx=(x-a_1)u(x-a_1);$$

$$\int (x-a_1)u(x-a_1)dx=\frac{(x-a_1)^2}{2}u(x-a_1).$$

and successively integrating Eq. (9), we obtain the following:

$$\frac{d^2\psi_x}{dx^2}=-\frac{q_1}{EJ}\left[\begin{array}{c}(x-a_1)u(x-a_1)-\\ (x-a_2)u(x-a_2)\end{array}\right]+C_1;$$

$$\frac{d\psi_x}{dx}=-\frac{q_1}{EJ}\left[\begin{array}{c}\frac{(x-a_1)^2}{2}u(x-a_1)-\\ \frac{(x-a_2)^2}{2}u(x-a_2)\end{array}\right]+C_1x+C_2;$$

$$\psi_x=-\frac{q_1}{EJ}\left[\begin{array}{c}\frac{(x-a_1)^3}{6}u(x-a_1)-\\ \frac{(x-a_2)^3}{6}u(x-a_2)\end{array}\right]+C_1\frac{x^2}{2}+C_2x+C_3.$$

By using boundary conditions (3), we will find C_1, C_2 and C_3:

$$C_3=0,C_1=\frac{q_1}{EJ}(a_2-a_1),$$

$$C_2=\frac{q_1}{EJ}\left[\frac{(L-a_1)^2}{2}-\frac{(L-a_2)^2}{2}-L(a_2-a_1)\right].$$

Based on the following condition:

$$Gh\left(\psi_x+\frac{\partial w}{\partial x}\right)=EJ\frac{d^2\psi_x}{dx^2}=-q_1\left[\begin{array}{c}(x-a_1)u(x-a_1)-\\ (x-a_2)u(x-a_2)-\\ a_2+a_1\end{array}\right],$$

we will get:

$$\frac{dw}{dx}=\frac{q_1}{EJ}\left[\frac{(x-a_1)^3}{6}u(x-a_1)-\frac{(x-a_2)^3}{6}u(x-a_2)\right]-C_1\frac{x^2}{2}-$$

$$C_2x-\frac{q_1}{Gh}\left[\begin{array}{c}(x-a_1)u(x-a_1)-\\ (x-a_2)u(x-a_2)-a_2+a_1\end{array}\right].$$

By integrating this expression, we will find the beam deflection function:

$$w(x)=\frac{q_1}{EJ}\left[\frac{(x-a_1)^4}{24}u(x-a_1)-\frac{(x-a_2)^4}{24}u(x-a_2)\right]-C_1\frac{x^3}{6}-C_2\frac{x^2}{2}-\frac{q_1}{Gh}\left[\frac{(x-a_1)^2}{2}u(x-a_1)-\frac{(x-a_2)^2}{2}u(x-a_2)-(a_2-a_1)x\right].$$

Based on the condition at x = 0, w = 0, we will get b_1=0.

In order to keep the expression for $w(x)$ as simple as possible, we will not substitute C_1 and C_2 with the values obtained.

If transverse shears are not considered, then:

$$w(x)=\frac{q_1}{EJ}\left[\frac{(x-a_1)^4}{24}u(x-a_1)-\frac{(x-a_2)^4}{24}u(x-a_2)\right]-C_1\frac{x^3}{6}-C_2\frac{x^2}{2}.$$

Therefore, with transverse shears considered, the deflection changed by Δ:

$$\Delta=-\frac{q_1}{Gh}\left[\frac{(x-a_1)^2}{2}u(x-a_1)-\frac{(x-a_2)^2}{2}u(x-a_2)-(a_2-a_1)x\right].$$

The maximum value will be at $x=L$:

$$\Delta_{max}=-\frac{q_1}{Gh}\left[\frac{(L-a_1)^2}{2}-\frac{(L-a_2)^2}{2}-(a_2-a_1)L\right].$$

Based on the condition $\frac{3q_1}{2Eh^2}\left(a_1^2-a_2^2\right)=0.05\cdot 0.0057$, we will estimate the L/h ratio. Let $a_1=\frac{L}{3}, a_2=\frac{2}{3}L$, then $\frac{q_1L^2}{2Eh^2}=0.05\cdot 0.0057.$ Hence, $\frac{L}{h}=29.$

Analysis results

The following table presents the analysis results:

Type of beam, with height *h* and span *L*	Type of load, uniformly distributed	Beam material	Recommended *L/h* ratio for beam analysis	
			by Kirchhoff model	by Timoshenko model
Cantilever	$0\le q\le L$	Concrete	≥ 16	<16
		Steel	≥ 60	<60
	$L/3\le q\le 2L/3$	Concrete	≥ 29	<29
Hinged support	$0\le q\le L$	Concrete	≥ 27	<27
		Steel	≥ 95	<95

Conclusion

When Bernoulli's hypothesis (Kirchhoff model) is applied, the relations used to determine the components of the stress-strain state of a beam resisting bending under various types of transverse load and beam end restraint are simple and clear as shown above. The obtained values of deflections and bending moments can be used in the analysis of rod systems, e.g., with the use of the displacement method. However, to ensure that analytical models and solutions are accurate, we need to evaluate the applicability of Bernoulli's hypothesis. This method makes it possible to do that easily. The obtained estimates for beams can be used approximately in the analysis of plates and shells.

References

ANSYS (2009). *Verification Manual for the Mechanical APDL Application*. Canonsburg, PA: ANSYS, Inc., 1924 p.

Cowper, G. R. (1966). The shear coefficient in Timoshenko's beam theory. *Journal of Applied Mechanics*, Vol. 33, Issue 2, pp. 335–340. DOI: 10.1115/1.3625046.

Donnell, L. H. (1982). *Beams, plates and shells*. Moscow: Nauka, 568 p.

Franco-Villafañe, J. A. and Méndez-Sánchez, R. A. (2016). On the accuracy of the Timoshenko beam theory above the critical frequency: best shear coefficient. *Journal of Mechanics*, Vol. 32, Issue 5, pp. 515–518. DOI: 10.1017/jmech.2015.104.

Goldenweiser, A. L. (1976). *Theory of thin elastic shells*. 2nd edition. Moscow: Nauka, 512 p.

Hutchinson, J. R. (1981). Transverse vibration of beams, exact versus approximate solutions. *Journal of Applied Mechanics*, Vol. 48, Issue 4, pp. 923–928. DOI: 10.1115/1.3157757.

Karpov, V., Kobelev, E., and Panin, A. (2021). Application of analytical solutions for bending beams in the method of movement. *Architecture and Engineering*, Vol. 6, No. 4, pp. 42–53. DOI: 10.23968/2500-0055-2021-6-4-42-53.

Lalin, V. V. and Beliaev, M. O. (2015). Bending of geometrically nonlinear cantilever beam. Results obtained by Cosserat – Timoshenko and Kirchhoff's rod theories. *Magazine of Civil Engineering*, No. 1, pp. 39–55. DOI: 10.5862/MCE.53.5.

Maslennikov, A. M. (2009). *Construction mechanics of rod systems: introductory course*. 2nd edition. Saint Petersburg: Prospekt Nauki, 240 p.

Méndez-Sáchez, R. A., Morales, A., and Flores, J. (2005). Experimental check on the accuracy of Timoshenko's beam theory. *Journal of Sound and Vibration*, Vol. 279, Issues 1–2, pp. 508–512. DOI: 10.1016/j.jsv.2004.01.050 .

Nazarov, S. A. (2002). *Asymptotic analysis of thin plates and rods*. Novosibirsk: Nauchnaya Kniga, 408 p.

Pavlenko, I. V. and Vereshchaka, S. M. (2002). Refined composite beam analysis using the Timoshenko model. *Scientific and Technical Conference of Lecturers, Employees, and Students of the Mechanics and Mathematics Department*. Sumy: Publishing House of Sumy State University, pp. 23–24.

Rossikhin, Yu. A. and Shitikova, M. V. (2010). Analytical review of the theories of Timoshenko type for thin-web beams of open profile. *Industrial and Civil Engineering*, No. 9, pp. 15–19.

Simulia (2012). *Abaqus 6.12. Benchmarks Manual*. Providence, RI: Dassault Systèmes Simulia Corp., 1738 p.

SOFiSTiK AG (2014). *Verification Manual.* Oberschleissheim: SOFiSTiK AG, 223 p.

Stephen, N. G. (1980). Timoshenko's shear coefficient from a beam subjected to gravity loading. *Journal of Applied Mechanics*, Vol. 47, Issue 1, pp. 121–127. DOI: 10.1115/1.3153589.

Timoshenko, S. P. (1945). *Strength of materials. Part 1. Elementary theory and problems*. Moscow, Leningrad; Gostekhizdat, 320 p.

Timoshenko, S. P. and Gere, J. M. (1976). *Mechanics of materials*. Moscow: Mir, 669 p.

Timoshenko, S. P. and Woinowsky-Krieger, S. (1963). *Plates and shells*. Moscow: Fizmatgiz, 636 p.

Tovstik, P. E. (2007). On the asymptotic nature of approximate models of beams, plates, and shells. *Vestnik St. Petersburg University, Mathematics*, Vol. 40, No. 3, pp. 188–192.

Yeliseyeva, O. P., Shadt, K. V., Timofeyeva, N. S., and Kononova, Ye. (2011). Application of the Timoshenko model in beam deflection analysis with account for bending and shear. *Challenging Issues of Aviation and Cosmonautics. Engineering Sciences*, pp. 91–93.

Zveryayev, Ye. M. (2003). Analysis of the hypotheses used when constructing the theory of beams and plates. *Journal of Applied Mathematics and Mechanics*, Vol. 67, Issue 3, pp. 425–434. DOI: 10.1016/S0021-8928(03)90026-8.

Zveryayev, Y. M. and Makarov, G. I. (2008). A general method for constructing Timoshenko-type theories. *Journal of Applied Mathematics and Mechanics*, Vol. 72, Issue 2, pp. 197–207. DOI: 10.1016/j.jappmathmech.2008.04.004.

ОЦЕНКА ПРИМЕНИМОСТИ ГИПОТЕЗЫ ПЛОСКИХ СЕЧЕНИЙ ПРИ РАСЧЕТЕ БАЛОК

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Аннотация
Используя свойства единичных функций, в данной статье находятся точные решения изгиба балки при различной поперечной нагрузке и различных условиях закрепления краев, как при использовании уравнений, основанных на гипотезе плоских сечений, так и на гипотезе, учитывающей поперечные сдвиги. Путем сравнения полученных аналитических решений для балки прямоугольного сечения находятся соотношения её длины L балки и ее высоты h, когда разница в прогибах меньше допустимой величины. Таким образом, получаются критерии использования гипотезы плоских сечений. Результаты расчета изгиба балок используются при исследовании стержневых систем методом сил и методом перемещений. При этом используется гипотеза плоских сечений. Все полученные соотношения имеют простой и наглядный вид. Однако эта гипотеза применима при расчете тонкостенных конструкций. А гипотеза, учитывающая поперечные сдвиги, может быть использована для конструкций средней высоты поперечного сечения. Для получения корректных результатов исследования строительных конструкций (балка, плита, оболочка, стержневая система) был необходим критерий применимости гипотезы плоских сечений (прямой нормали). **Цель исследования:** Построение математической модели деформирования и создание методики расчета на изгиб упругих балок типа Тимошенко с учетом поперечных сдвигов. **Методы:** На основе применения математического аппарата обобщенных функций методом непосредственного интегрирования дифференциального уравнения изогнутой оси балки получены аналитические выражения функции прогибов для различных граничных условий. **Результаты:** По предложенной методике проведены расчеты балок при действии различной поперечной нагрузки и различных видов закрепления концов краев балки. Выполнена оценка области применения гипотезы плоских сечений для основных типов балок, используемых для расчетов стержневых систем методом перемещений.

Ключевые слова
Балка, изгиб, модель Кирхгофа, поперечный сдвиг, модель Тимошенко, единичные функции.