VARIATIONAL METHOD FOR DERIVATION OF EQUATIONS OF MIXED TYPE FOR SHELLS OF A GENERAL TYPE

Vladimir Karpov

Saint Petersburg State University of Architecture and Civil Engineering,
Vtoraja Krasnoarmejskaja ul. 4, St. Petersburg, 190005, Russia.
vvkarpov@lan.spbgasu.ru

Abstract

In this work, derivation of equations of a mixed type for shallow shell constructions of an arbitrary type is carried out by means of the variational method. Such equations are more simplified equations of the shell theory, as compared to equations in displacements, but in case of some types of fixing of shell edges (for example, in case of pin-edge and movable fixing) they are more convenient. The mathematical model of shell deformation is based on the Kirchhoff–Love hypotheses, geometrical nonlinearity is taken into consideration.

The full functional of shell energy is used for derivation of equilibrium equations and the third equation of strain compatibility in the middle surface of a shell, its minimum condition (the first variation of the functional has to be equal to zero) giving place to these equations. The stress function is entered in the middle surface of the shell in such a way as to make the first two equilibrium equations vanish identically. Thus, the third equilibrium equation and the equation of strain compatibility give the equation of a mixed type in relation to the deflection function and the stress function in the middle surface.

Key words: shells, mathematical model, equations of a mixed type, variational method.

Introduction

Intensive development of the nonlinear shell theory begins in the 1930/40s. Primarily, it concerns shallow shells (Kh.M. Mushtari (1939), L.N. Donell (1934), V.Z. Vlasov (1949)).

With the advent of computer in the early 1960s, geometrical nonlinearity started to be considered in shell stability analysis. For shallow shells of right-angular design, the analysis was carried out mainly on the basis of equations of a mixed type (V.Z. Vlasov (1949), V.V. Petrov (1975), V.A. Krysko (1976), etc.), which are still used quite often (Nikitin, Stupishin, and Vatanin, 2012; Kolomoets and Modin, 2014; Spasskaya and Treshchev, 2015; Shen S-H. and Yang D-Q., 2014; Zhang J. and van Campen D.H., 2003; van Campen et al., 2002; Seffen, 2007; Karpov, 2010, etc.). Shells of uniform thickness, for which equations of a mixed type were obtained by V. Z. Vlasov, were mainly considered.

Equations of a mixed type are derived only for shallow shells; nevertheless, they are widely used in construction, as it is easy to choose approximating functions for such equations in case of pin-edge and movable fixing of the shell contour. Such a form of fixing allows to avoid stress concentration near the shell contour, but it is difficult to choose approximating functions for it when using equations in displacements.

In this work, equations of a mixed type for shells of an arbitrary type (but shallow) are derived. The third equilibrium equation and one of the equations of strain compatibility are used for derivation of these equations in the middle surface of the shell. Equations of strain compatibility for shells of a general type were obtained for the first time by A.L. Goldenveyzer (1940) by means of formulation of the Gauss–Codazzi conditions for the deformed middle surface.

Subject matter, tasks and methods

The objective of this work is derivation of equations of a mixed type for shallow shell constructions of an arbitrary type.

Findings and discussion

Equations of a mixed type are more simplified equations of the shell theory, as compared to equations in displacements, but in case of some types of fixing of shell edges (for example, in case of pin-edge and movable fixing) they are more convenient.

Now, we will obtain equations of a mixed type, which represent a system of two differential equations in relation to normal displacement $W = W(x, y)$ and the stress function in the middle surface $\Phi = \Phi(x, y)$.

Let the normal static loading , which is rather uniformly distributed across the surface, influence the shell. Let us believe that the shell is either shallow or splits into shallow parts in the process of deformation. The middle surface of the shell in thickness is taken as a coordinate surface. The *x*- and *y*-axes of the orthogonal coordinate system are directed along the lines of principal curvatures of the shell, the *z*-axis is directed orthogonally to the coordinate surface towards concavity.

Let us enter the stress function $\Phi(x, y)$, connected with the forces by the following dependences (Pertsev and Platonov, 1987):

$$N_x = \frac{1}{B}\frac{\partial}{\partial y}\left(\frac{1}{B}\frac{\partial \Phi}{\partial y}\right) + \frac{1}{A^2 B}\frac{\partial B}{\partial x}\frac{\partial \Phi}{\partial x} = F_1(\Phi),$$

$$N_y = \frac{1}{A}\frac{\partial}{\partial x}\left(\frac{1}{A}\frac{\partial \Phi}{\partial x}\right) + \frac{1}{AB^2}\frac{\partial A}{\partial y}\frac{\partial \Phi}{\partial y} = F_2(\Phi), \qquad (1)$$

$$N_{xy} = \frac{1}{AB}\left(\frac{1}{A}\frac{\partial A}{\partial y}\frac{\partial \Phi}{\partial x} + \frac{1}{B}\frac{\partial B}{\partial x}\frac{\partial \Phi}{\partial y} - \frac{\partial^2 \Phi}{\partial x \partial y}\right) = F_3(\Phi).$$

One of the methods for obtaining equations of a mixed type is as follows:

If we substitute (1) in the first two equilibrium equations, these equations will be identically satisfied. And the third equilibrium equation

$$AB\left(k_x N_x + k_y N_y\right) - \frac{\partial}{\partial x} B\left(N_x \theta_1 + N_{xy}\theta_2\right) - \frac{\partial}{\partial y} A\left(N_y \theta_2 + N_{xy}\theta_1\right) +$$
$$+\frac{\partial}{\partial x}\left[\frac{1}{A}\left(\frac{\partial BM_x}{\partial x} - M_y\frac{\partial B}{\partial x} + \frac{\partial AM_{xy}}{\partial y} + M_{xy}\frac{\partial A}{\partial y}\right)\right] +$$
$$+\frac{\partial}{\partial y}\left[\frac{1}{B}\left(\frac{\partial AM_y}{\partial y} - M_x\frac{\partial A}{\partial y} + \frac{\partial BM_{xy}}{\partial x} + M_{xy}\frac{\partial B}{\partial x}\right)\right] + ABq = 0$$

will give the first equation of a mixed type after replacement of the forces N_x, N_y, N_{xy} through the function $\Phi(x, y)$ according to rule (1). In addition, the moments are expressed through the function $W(x, y)$ and

$$\theta_1 = -\frac{1}{A}\frac{\partial W}{\partial x},\ \theta_2 = -\frac{1}{B}\frac{\partial W}{\partial y};$$

here, A,B — Lame parameters.

The second equation of the system is found by means of the third equation of strain compatibility:

$$-AB\left(k_x\chi_2 + k_y\chi_1 - \chi_1\chi_2 + \chi_{12}^2\right) =$$
$$= \frac{\partial}{\partial y}\left[\frac{1}{B}\left(\frac{\partial B}{\partial x}\gamma_{xy} + \frac{1}{2}B\frac{\partial \gamma_{xy}}{\partial x} + \frac{\partial A}{\partial y}\varepsilon_y - \frac{\partial A\varepsilon_x}{\partial y}\right)\right] +$$
$$+\frac{\partial}{\partial x}\left[\frac{1}{A}\left(\frac{\partial A}{\partial y}\gamma_{xy} + \frac{1}{2}A\frac{\partial \gamma_{xy}}{\partial y} + \frac{\partial B}{\partial x}\varepsilon_x - \frac{\partial B\varepsilon_y}{\partial x}\right)\right].$$

Here,

$$\chi_1 = \frac{1}{A}\frac{\partial \theta_1}{\partial x} + \frac{1}{AB}\frac{\partial A}{\partial y}\theta_2,\ \chi_2 = \frac{1}{B}\frac{\partial \theta_2}{\partial y} + \frac{1}{AB}\frac{\partial B}{\partial x}\theta_1,$$

$$2\chi_{12} = \frac{1}{A}\frac{\partial \theta_2}{\partial x} + \frac{1}{B}\frac{\partial \theta_1}{\partial y} - \frac{1}{AB}\left(\frac{\partial A}{\partial y}\theta_1 + \frac{\partial B}{\partial x}\theta_2\right).$$

From the ratios

$$N_x = \frac{Eh}{1-\mu^2}\left(\varepsilon_x + \mu\varepsilon_y\right),\quad N_y = \frac{Eh}{1-\mu^2}\left(\varepsilon_y + \mu\varepsilon_x\right),$$

$$N_{xy} = \frac{Eh}{2(1+\mu)}\gamma_{xy}$$

we will express deformations through forces and replace forces in them by the expressions (1)

$$\tilde{\varepsilon}_x = \frac{1}{Eh}\left(N_x - \mu N_y\right) = \frac{1}{Eh}\left(F_1(\Phi) - \mu F_2(\Phi)\right),$$

$$\tilde{\varepsilon}_y = \frac{1}{Eh}\left(N_y - \mu N_x\right) = \frac{1}{Eh}\left(F_2(\Phi) - \mu F_1(\Phi)\right), \qquad (2)$$

$$\tilde{\gamma}_{xy} = \frac{2(1+\mu)}{Eh}N_{xy} = \frac{2(1+\mu)}{Eh}F_3(\Phi).$$

Here, *E*, μ — modulus of elasticity and Poisson's ratio of the shell material.

Further, we will substitute the found expressions of deformations in the third equation of strain compatibility.

Let us consider the variational method for derivation of equations of a mixed type (Karpov, 2010) to substantiate the considered method for derivation of equations of a mixed type and derivation of the third equation of strain compatibility.

Equations of a mixed type can be obtained from the minimum condition of the full functional (Abovskiy, Andreev, and Deruga, 1978), which can be written in the following form

$$E_p = \frac{1}{2}\int_0^a\int_0^b \{2N_x\varepsilon_x - N_x\tilde{\varepsilon}_x + 2N_y\varepsilon_y - N_y\tilde{\varepsilon}_y + 2N_{xy}\gamma_{xy} - N_{xy}\tilde{\gamma}_{xy}$$
$$+2M_x\chi_1 - M_x\tilde{\chi}_1 + 2M_y\chi_2 - M_y\tilde{\chi}_2 + \qquad (3)$$
$$+2M_{xy}2\chi_{12} - M_{xy}2\tilde{\chi}_{12} - 2qW\}\, ABdxdy.$$

Here, $\varepsilon_x, \varepsilon_y, \gamma_{xy}$ — deformations expressed through displacements, and $\tilde{\varepsilon}_x, \tilde{\varepsilon}_y, \tilde{\gamma}_{xy}$ — deformations expressed through forces (2); $\chi_1, \chi_2, \chi_{12}$ — deformations connected with bending and torsion, expressed through deflection , and $\tilde{\chi}_1, \tilde{\chi}_2, \tilde{\chi}_{12}$ — deformations expressed through moments, but they coincide for shells of uniform thickness respectively with $\chi_1, \chi_2, \chi_{12}$.

Finding the first variation of the functional (3) and setting it equal to zero, we will obtain

$$\delta E_p = \frac{1}{2}\int_0^a\int_0^b \left\{2N_x\delta\left[B\frac{\partial U}{\partial x} + \frac{\partial A}{\partial y}V - ABk_x W + \frac{1}{2}AB\theta_1^2\right] +\right.$$
$$+2\varepsilon_x\delta\left[A\frac{\partial}{\partial y}\left(\frac{1}{B}\frac{\partial \Phi}{\partial y}\right) + \frac{1}{A}\frac{\partial B}{\partial x}\frac{\partial \Phi}{\partial x}\right] -$$

$$-\frac{1}{Eh}N_x\delta\Bigg[A\frac{\partial}{\partial y}\left(\frac{1}{B}\frac{\partial\Phi}{\partial y}\right)+$$

$$+\frac{1}{A}\frac{\partial B}{\partial x}\frac{\partial\Phi}{\partial x}-\mu B\frac{\partial}{\partial x}\left(\frac{1}{A}\frac{\partial\Phi}{\partial x}\right)-\mu\frac{1}{B}\frac{\partial A}{\partial y}\frac{\partial\Phi}{\partial y}\Bigg]-$$

$$-\tilde{\varepsilon}_x\delta\left[A\frac{\partial}{\partial y}\left(\frac{1}{B}\frac{\partial\Phi}{\partial y}\right)+\frac{1}{A}\frac{\partial B}{\partial x}\frac{\partial\Phi}{\partial x}\right]+$$

$$+2N_y\delta\left[A\frac{\partial V}{\partial y}+\frac{\partial B}{\partial x}U-ABk_yW+\frac{1}{2}AB\theta_2^2\right]+$$

$$+2\varepsilon_y\delta\left[B\frac{\partial}{\partial x}\left(\frac{1}{A}\frac{\partial\Phi}{\partial x}\right)+\frac{1}{B}\frac{\partial A}{\partial y}\frac{\partial\Phi}{\partial y}\right]-$$

$$-\frac{1}{Eh}N_y\delta\Bigg[B\frac{\partial}{\partial x}\left(\frac{1}{A}\frac{\partial\Phi}{\partial x}\right)+$$

$$+\frac{1}{B}\frac{\partial A}{\partial y}\frac{\partial\Phi}{\partial y}-\mu A\frac{\partial}{\partial y}\left(\frac{1}{B}\frac{\partial\Phi}{\partial y}\right)-\mu\frac{1}{A}\frac{\partial B}{\partial x}\frac{\partial\Phi}{\partial x}\Bigg]-$$

$$-\tilde{\varepsilon}_y\delta\left[B\frac{\partial}{\partial x}\left(\frac{1}{A}\frac{\partial\Phi}{\partial x}\right)+\frac{1}{B}\frac{\partial A}{\partial y}\frac{\partial\Phi}{\partial y}\right]+$$

$$+2N_{xy}\delta\left[B\frac{\partial V}{\partial x}+A\frac{\partial U}{\partial y}-\frac{\partial A}{\partial y}U-\frac{\partial B}{\partial x}V+AB\theta_1\theta_2\right]+$$

$$+2\gamma_{xy}\delta\left[\frac{1}{A}\frac{\partial A}{\partial y}\frac{\partial\Phi}{\partial x}+\frac{1}{B}\frac{\partial B}{\partial x}\frac{\partial\Phi}{\partial y}-\frac{\partial^2\Phi}{\partial x\partial y}\right]-$$

$$-2\frac{(1+\mu)}{Eh}N_{xy}\delta\left[\frac{1}{A}\frac{\partial A}{\partial y}\frac{\partial\Phi}{\partial x}+\frac{1}{B}\frac{\partial B}{\partial x}\frac{\partial\Phi}{\partial y}-\frac{\partial^2\Phi}{\partial x\partial y}\right]-$$

$$-\tilde{\gamma}_{xy}\delta\left[\frac{1}{A}\frac{\partial A}{\partial y}\frac{\partial\Phi}{\partial x}+\frac{1}{B}\frac{\partial B}{\partial x}\frac{\partial\Phi}{\partial y}-\frac{\partial^2\Phi}{\partial x\partial y}\right]+$$

$$+D(\chi_1+\mu\chi_2)\delta\left[B\frac{\partial\theta_1}{\partial x}+\frac{\partial A}{\partial y}\theta_2\right]+$$

$$+D\chi_1\delta\left[B\frac{\partial\theta_1}{\partial x}+\frac{\partial A}{\partial y}\theta_2+\mu A\frac{\partial\theta_2}{\partial y}+\mu\frac{\partial B}{\partial x}\theta_1\right]+$$

$$+D(\chi_2+\mu\chi_1)\delta\left[A\frac{\partial\theta_2}{\partial y}+\frac{\partial B}{\partial x}\theta_1\right]+$$

$$+D\chi_2\delta\left[A\frac{\partial\theta_2}{\partial y}+\frac{\partial B}{\partial x}\theta_1+\mu B\frac{\partial\theta_1}{\partial x}+\mu\frac{\partial A}{\partial y}\theta_2\right]+$$

$$+2D\bar{\mu}2\chi_{12}\delta\left[B\frac{\partial\theta_2}{\partial x}+A\frac{\partial\theta_1}{\partial y}-\frac{\partial A}{\partial y}\theta_1-\frac{\partial B}{\partial x}\theta_2\right]+$$

$$-2ABq\delta W\Big\}\,dxdy=0,$$

where $D=\dfrac{Eh^3}{12\left(1-\mu^2\right)}$, $\bar{\mu}=0,5(1-\mu)$.

Let us transform the variational equation in such a way that there are no variations from derived functions under the double integral sign. As a result, we will obtain

$$-\int_0^a\!\!\int_0^b\Bigg\{2\left[\frac{\partial BN_x}{\partial x}-\frac{\partial B}{\partial x}N_y+\frac{\partial AN_{xy}}{\partial y}+N_{xy}\frac{\partial A}{\partial y}\right]\delta U+$$

$$+2\left[\frac{\partial AN_y}{\partial y}-\frac{\partial A}{\partial y}N_x+\frac{\partial BN_{xy}}{\partial x}+N_{xy}\frac{\partial B}{\partial x}\right]\delta V+$$

$$+\Bigg[\underline{-\frac{\partial}{\partial y}\Bigg(2\frac{1}{B}\frac{\partial A\varepsilon_x}{\partial y}-\frac{1}{B}\frac{\partial A\tilde{\varepsilon}_x}{\partial y}-2\frac{1}{B}\frac{\partial A}{\partial y}\varepsilon_y+}$$

$$\underline{+\frac{1}{B}\frac{\partial A}{\partial y}\tilde{\varepsilon}_y-2\frac{1}{B}\frac{\partial B}{\partial x}\gamma_{xy}+\frac{1}{B}\frac{\partial B}{\partial x}\tilde{\gamma}_{xy}-\frac{\partial\gamma_{xy}}{\partial x}+\frac{1}{2}\frac{\partial\tilde{\gamma}_{xy}}{\partial x}\Bigg)-}$$

$$\underline{-\frac{\partial}{\partial x}\Bigg(2\frac{1}{A}\frac{\partial B\varepsilon_y}{\partial x}-\frac{1}{A}\frac{\partial B\tilde{\varepsilon}_y}{\partial x}-2\frac{1}{A}\frac{\partial B}{\partial x}\varepsilon_x+\frac{1}{A}\frac{\partial B}{\partial x}\tilde{\varepsilon}_x-}$$

$$\underline{-2\frac{1}{A}\frac{\partial A}{\partial y}\gamma_{xy}+\frac{1}{A}\frac{\partial A}{\partial y}\tilde{\gamma}_{xy}-\frac{\partial\gamma_{xy}}{\partial y}+\frac{1}{2}\frac{\partial\tilde{\gamma}_{xy}}{\partial y}\Bigg)}+$$

$$+\frac{1}{Eh}\Bigg(\frac{\partial}{\partial y}\frac{1}{B}\Bigg(\frac{\partial AN_x}{\partial y}+\mu\frac{\partial A}{\partial y}N_x-\frac{\partial A}{\partial y}N_y-$$

$$-\mu\frac{\partial AN_y}{\partial y}-2(1+\mu)\frac{\partial B}{\partial x}N_{xy}-(1+\mu)B\frac{\partial N_{xy}}{\partial x}\Bigg)+$$

$$+\frac{\partial}{\partial x}\frac{1}{A}\Bigg(\frac{\partial BN_y}{\partial x}+\mu\frac{\partial B}{\partial x}N_y-\frac{\partial B}{\partial x}N_x-\mu\frac{\partial BN_x}{\partial x}-$$

$$-2(1+\mu)\frac{\partial A}{\partial y}N_{xy}-(1+\mu)A\frac{\partial N_{xy}}{\partial y}\Bigg)\Bigg)\Bigg]\delta\Phi+$$

$$+2\Bigg[AB\left(k_xN_x+k_yN_y\right)-\frac{\partial}{\partial x}B\left(N_x\theta_1+N_{xy}\theta_2\right)-$$

$$-\frac{\partial}{\partial y}A\left(N_y\theta_2+N_{xy}\theta_1\right)+$$

$$+D\Bigg(\frac{\partial}{\partial x}\frac{1}{A}\Bigg(\frac{\partial}{\partial x}B(\chi_1+\mu\chi_2)-$$

$$-\frac{\partial B}{\partial x}(\chi_2+\mu\chi_1)+\bar{\mu}\frac{\partial}{\partial y}(A2\chi_{12})+\bar{\mu}\frac{\partial A}{\partial y}2\chi_{12}\Bigg)+$$

$$+\frac{\partial}{\partial y}\frac{1}{B}\Bigg(\frac{\partial}{\partial y}A(\chi_2+\mu\chi_1)-\frac{\partial A}{\partial y}(\chi_1+\mu\chi_2)+$$

$$+\bar{\mu}\frac{\partial}{\partial x}(B2\chi_{12})+\bar{\mu}\frac{\partial B}{\partial x}2\chi_{12}\Bigg)\Bigg)+ABq\Bigg]\delta W\Bigg\}dxdy+$$

$$+\int_0^b\Bigg\{2BN_x\delta U+2BN_{xy}\delta V+\frac{1}{A}\Bigg[2\frac{\partial B}{\partial x}\varepsilon_x-\frac{\partial B}{\partial x}\tilde{\varepsilon}_x-$$

$$-2\frac{\partial B\varepsilon_y}{\partial x}+\frac{\partial B\tilde{\varepsilon}_y}{\partial x}+2\frac{\partial A}{\partial y}\gamma_{xy}-\frac{\partial A}{\partial y}\tilde{\gamma}_{xy}+2A\frac{\partial\gamma_{xy}}{\partial y}-$$

$$-A\frac{\partial\tilde{\gamma}_{xy}}{\partial y}-\frac{1}{Eh}\Bigg(\mu\frac{\partial BN_x}{\partial x}+\frac{\partial B}{\partial x}N_x-\frac{\partial BN_y}{\partial x}-\mu\frac{\partial B}{\partial x}N_y+$$

$$+2(1+\mu)A\frac{\partial N_{xy}}{\partial y}+2(1+\mu)\frac{\partial A}{\partial y}N_{xy}\Bigg)\Bigg]\delta\Phi+$$

$$+\frac{B}{A}\left[2\varepsilon_y-\tilde{\varepsilon}_y+\frac{1}{Eh}\left(\mu N_x-N_y\right)\right]\delta\frac{\partial\Phi}{\partial x}+$$

$$+2\Bigg[-BN_x\theta_1-BN_{xy}\theta_2+D\frac{1}{A}\Bigg(\frac{\partial}{\partial x}B(\chi_1+\mu\chi_2)-$$

$$-\frac{\partial B}{\partial x}(\chi_2+\mu\chi_1)+2\bar{\mu}\frac{\partial A2\chi_{12}}{\partial y}\Bigg)\Bigg]\delta W-$$

$$-2D\frac{B}{A}(\chi_1+\mu\chi_2)\delta\frac{\partial W}{\partial x}\Bigg\}\Bigg|_{x=0}^{x=a}dy+$$

$$+\int_0^a \left\{ 2AN_{xy}\delta U + 2AN_y\delta V + \frac{1}{B}\left[-2\frac{\partial A\varepsilon_x}{\partial y} + \frac{\partial A\tilde{\varepsilon}_x}{\partial y} + \right.\right.$$

$$+2\frac{\partial A}{\partial y}\varepsilon_y - \frac{\partial A}{\partial y}\tilde{\varepsilon}_y + 2\frac{\partial B}{\partial x}\gamma_{xy} - \frac{\partial B}{\partial x}\tilde{\gamma}_{xy} + 2B\frac{\partial \gamma_{xy}}{\partial x} -$$

$$-B\frac{\partial \tilde{\gamma}_{xy}}{\partial x} + \frac{1}{Eh}\left(\frac{\partial AN_x}{\partial y} + \mu\frac{\partial A}{\partial y}N_x - \frac{\partial A}{\partial y}N_y - \mu\frac{\partial AN_y}{\partial y} - \right.$$

$$\left.\left. -2(1+\mu)\frac{\partial B}{\partial x}N_{xy} - 2(1+\mu)B\frac{\partial N_{xy}}{\partial x}\right)\right]\delta\Phi +$$

$$+\frac{A}{B}\left[2\varepsilon_x - \tilde{\varepsilon}_x - \frac{1}{Eh}\left(N_x - \mu N_y\right)\right]\delta\frac{\partial \Phi}{\partial y} +$$

$$+2\left[-AN_y\theta_2 - AN_{xy}\theta_1 + D\frac{1}{B}\left(\frac{\partial}{\partial y}A\left(\chi_2 + \mu\chi_1\right) - \right.\right.$$

$$\left.\left. -\frac{\partial A}{\partial y}\left(\chi_1 + \mu\chi_2\right) + 2\bar{\mu}\frac{\partial B2\chi_{12}}{\partial x}\right)\right]\delta W -$$

$$\left. -2D\frac{A}{B}\left(\chi_2 + \mu\chi_1\right)\delta\frac{\partial W}{\partial y}\right\}\Bigg|_{y=0}^{y=b} dx -$$

$$-\left[2\gamma_{xy} - \tilde{\gamma}_{xy} - \frac{2(1+\mu)}{Eh}N_{xy}\right]\delta\Phi\Bigg|_{x=0}^{x=a}\Bigg|_{y=0}^{y=b} -$$

$$-4\bar{\mu}D2\chi_{12}\delta W\Big|_{x=0}^{x=a}\Big|_{y=0}^{y=b} = 0.$$

Factors at δU and δV in the double integral have to be equal to zero and these are the 1st and the 2nd equilibrium equations.

If we carry out some analysis of the obtained variational equation, we will notice that $\varepsilon_x, \varepsilon_y, \gamma_{xy}$ and $\tilde{\varepsilon}_x, \tilde{\varepsilon}_y, \tilde{\gamma}_{xy}$ are the same deformations, only having different expressions, thus $\varepsilon_x - \tilde{\varepsilon}_x = 0$, $\varepsilon_y - \tilde{\varepsilon}_y = 0$, $\gamma_{xy} - \tilde{\gamma}_{xy} = 0$. Therefore, factors at $\delta\Phi$, $\delta\frac{\partial\Phi}{\partial x}$, $\delta\frac{\partial\Phi}{\partial y}$ will be identically equal to zero in one-dimensional integrals.

Now, let us analyze factors in a double integral at $\delta\Phi$. The underlined expression can be written in the following form

$$-\frac{\partial}{\partial y}\left[\frac{1}{B}\left(\frac{\partial A\varepsilon_x}{\partial y} - \frac{\partial A}{\partial y}\varepsilon_y - \frac{\partial B}{\partial x}\gamma_{xy} - \frac{1}{2}B\frac{\partial \gamma_{xy}}{\partial x}\right)\right] -$$

$$-\frac{\partial}{\partial x}\left[\frac{1}{A}\left(\frac{\partial B\varepsilon_y}{\partial x} - \frac{\partial B}{\partial x}\varepsilon_x - \frac{\partial A}{\partial y}\gamma_{xy} - \frac{1}{2}A\frac{\partial \gamma_{xy}}{\partial y}\right)\right].$$

This expression represents the right part of the third equation of strain compatibility. If we substitute the expression of deformation through displacements in it, the terms of the equation containing *U* and *V* will be mutually reduced and it will be as follows

$$-AB\left(k_x\chi_2 + k_y\chi_1 - \chi_1\chi_2 + \chi_{12}^2\right).$$

The expression, not underlined and standing as a factor at $\delta\Phi$. in the double integral, can be transformed:

$$\frac{\partial}{\partial y}\frac{1}{B}\left(\frac{\partial A\tilde{\varepsilon}_x}{\partial y} - \frac{\partial A}{\partial y}\tilde{\varepsilon}_y - \frac{\partial B}{\partial x}\tilde{\gamma}_{xy} - \frac{1}{2}B\frac{\partial \tilde{\gamma}_{xy}}{\partial x}\right) +$$

$$+\frac{\partial}{\partial x}\frac{1}{A}\left(\frac{\partial B\tilde{\varepsilon}_y}{\partial x} - \frac{\partial B}{\partial x}\tilde{\varepsilon}_x - \frac{\partial A}{\partial y}\tilde{\gamma}_{xy} - \frac{1}{2}A\frac{\partial \tilde{\gamma}_{xy}}{\partial y}\right).$$

Thus, the factor at $\delta\Phi$. in the double integral will be as follows

$$2\left[\frac{\partial}{\partial y}\frac{1}{B}\left(\frac{\partial A\tilde{\varepsilon}_x}{\partial y} - \frac{\partial A}{\partial y}\tilde{\varepsilon}_y - \frac{\partial B}{\partial x}\tilde{\gamma}_{xy} - \frac{1}{2}B\frac{\partial \tilde{\gamma}_{xy}}{\partial x}\right) + \right.$$

$$+\frac{\partial}{\partial x}\frac{1}{A}\left(\frac{\partial B\tilde{\varepsilon}_y}{\partial x} - \frac{\partial B}{\partial x}\tilde{\varepsilon}_x - \frac{\partial A}{\partial y}\tilde{\gamma}_{xy} - \frac{1}{2}A\frac{\partial \tilde{\gamma}_{xy}}{\partial y}\right) -$$

$$-\frac{\partial}{\partial y}\frac{1}{B}\left(\frac{\partial A\varepsilon_x}{\partial y} - \frac{\partial A}{\partial y}\varepsilon_y - \frac{\partial B}{\partial x}\gamma_{xy} - \frac{1}{2}B\frac{\partial \gamma_{xy}}{\partial x}\right) -$$

$$\left. -\frac{\partial}{\partial x}\frac{1}{A}\left(\frac{\partial B\varepsilon_y}{\partial x} - \frac{\partial B}{\partial x}\varepsilon_x - \frac{\partial A}{\partial y}\gamma_{xy} - \frac{1}{2}A\frac{\partial \gamma_{xy}}{\partial y}\right)\right]\delta\Phi.$$

On the basis of the equation of strain compatibility, the last two summands are equal to

$$-AB\left(k_x\chi_2 + k_y\chi_1 - \chi_1\chi_2 + \chi_{12}^2\right).$$

As the factor before $\delta\Phi$. in the double integral shall vanish, we will obtain

$$\frac{\partial}{\partial y}\frac{1}{B}\left(\frac{\partial A\tilde{\varepsilon}_x}{\partial y} - \frac{\partial A}{\partial y}\tilde{\varepsilon}_y - \frac{\partial B}{\partial x}\tilde{\gamma}_{xy} - \frac{1}{2}B\frac{\partial \tilde{\gamma}_{xy}}{\partial x}\right) +$$

$$+\frac{\partial}{\partial x}\frac{1}{A}\left(\frac{\partial B\tilde{\varepsilon}_y}{\partial x} - \frac{\partial B}{\partial x}\tilde{\varepsilon}_x - \frac{\partial A}{\partial y}\tilde{\gamma}_{xy} - \frac{1}{2}A\frac{\partial \tilde{\gamma}_{xy}}{\partial y}\right) =$$

$$= AB\left(k_x\chi_2 + k_y\chi_1 - \chi_1\chi_2 + \chi_{12}^2\right).$$

On the one hand, the third equation of strain compatibility is obtained by the variational method. On the other hand, having replaced $\tilde{\varepsilon}_x, \tilde{\varepsilon}_y, \tilde{\gamma}_{xy}$ by their expressions (2), we will obtain one of the equations of a mixed type. The second equation of a mixed type is obtained by means of setting the factor at δW in the double integral equal to zero and replacement of the forces by expressions (1) in it:

$$AB\left(k_xF_1(\Phi) + k_yF_2(\Phi)\right) - \frac{\partial}{\partial x}B\left(F_1(\Phi)\theta_1 + F_3(\Phi)\theta_2\right) -$$

$$-\frac{\partial}{\partial y}A\left(F_2(\Phi)\theta_2 + F_3(\Phi)\theta_1\right) +$$

$$+D\left[\frac{\partial}{\partial x}\frac{1}{A}\left(\frac{\partial}{\partial x}B\left(\chi_1 + \mu\chi_2\right) - \frac{\partial B}{\partial x}\left(\chi_2 + \mu\chi_1\right) + \right.\right.$$

$$\left. +\bar{\mu}\frac{\partial}{\partial y}\left(A2\chi_{12}\right) + \bar{\mu}\frac{\partial A}{\partial y}2\chi_{12}\right) +$$

$$+\frac{\partial}{\partial y}\frac{1}{B}\left(\frac{\partial}{\partial y}A\left(\chi_2 + \mu\chi_1\right) - \frac{\partial A}{\partial y}\left(\chi_1 + \mu\chi_2\right) + \right.$$

$$\left.\left. +\bar{\mu}\frac{\partial}{\partial x}\left(B2\chi_{12}\right) + \bar{\mu}\frac{\partial B}{\partial x}2\chi_{12}\right)\right] + ABq = 0, \quad (4)$$

$$-AB\left(k_x\chi_2 + k_y\chi_1 - \chi_1\chi_2 + \chi_{12}^2\right) +$$

$$+\frac{1}{Eh}\left[\frac{\partial}{\partial y}\frac{1}{B}\left(\frac{\partial AF_1(\Phi)}{\partial y}-\frac{\partial A}{\partial y}F_2(\Phi)-\frac{\partial BF_3(\Phi)}{\partial x}-\frac{\partial B}{\partial x}F_3(\Phi)\right)+\right.$$
$$\left.+\frac{\partial}{\partial x}\frac{1}{A}\left(\frac{\partial BF_2(\Phi)}{\partial x}-\frac{\partial B}{\partial x}F_1(\Phi)-\frac{\partial AF_3(\Phi)}{\partial y}-\frac{\partial A}{\partial y}F_3(\Phi)\right)\right]=0.$$

If shallow shells of right-angled design (*A = 1, B = 1*) are considered, the equations (4) are written as

$$D\Delta\Delta W = L(W,\Phi)+\nabla_k^2\Phi+q,$$
$$\frac{1}{Eh}\Delta\Delta\Phi+\frac{1}{2}L(W,W)+\nabla_k^2 W=0,$$

where

$$\Delta=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2},\ \nabla_k^2 A_1=k_x\frac{\partial^2 A_1}{\partial y^2}+k_y\frac{\partial^2 A_1}{\partial x^2},$$
$$L(A_1,A_2)=\frac{\partial^2 A_1}{\partial x^2}\frac{\partial^2 A_2}{\partial y^2}+\frac{\partial^2 A_1}{\partial y^2}\frac{\partial^2 A_2}{\partial x^2}-2\frac{\partial^2 A_1}{\partial x\partial y}\frac{\partial^2 A_2}{\partial x\partial y}.$$

The system of equations (4) is a system of differential equations in partial derivatives of the eighth order in relation to the required functions $W(x,y)$ and $\Phi(x,y)$ (each of these functions contains derivatives up to and including the fourth order on variables *x* and *y*). For solution of this system, it is necessary to set four edge conditions on each edge of the shell.

It is possible to write edge conditions (factors standing before $\delta\Phi,\ \delta\frac{\delta\Phi}{\delta x},\delta\frac{\delta\Phi}{\delta y}$ are identically equal to zero)

at $x=0,\ x=a$

$U=0$ or $N_x=0,\ V=0$ or $N_{xy}=0,$

$$W=0 \text{ or } \frac{1}{AB}\left(\frac{\partial BM_x}{\partial x}-\frac{\partial B}{\partial x}M_y+2\frac{\partial AM_{xy}}{\partial y}\right)-N_x\theta_1-N_{xy}\theta_2=0,$$

$\frac{\partial W}{\partial x}=0$ or $M_x=0$;

at $y=0,\ y=b$

$U=0$ or $N_{xy}=0, V=0$ or $N_y=0,$

$$W=0 \text{ or } \frac{1}{AB}\left(\frac{\partial AM_y}{\partial y}-\frac{\partial A}{\partial y}M_x+2\frac{\partial BM_{xy}}{\partial x}\right)-N_y\theta_2-N_{xy}\theta_1=0,$$

$\frac{\partial W}{\partial y}=0$ or $M_y=0$

from the equality to zero of one-dimensional integrals in the variational equation obtained after transformation, having reduced factors standing before $\delta U,\delta V,\ \delta W,\delta\frac{\partial W}{\partial x},\delta\frac{\partial W}{\partial y}$, to the form containing forces and moments.

Besides, we have $M_{xy}=0$ or $W=0$ in the angular points of the shell contour at $x=0,\ x=a$ and $y=0$ or $y=b$.

Equations of a mixed type are often applied in case of pin-edge and movable fixing of the shell contour.

In this case, for example, at *x = 0, x=a U* ≠ 0 therefore it shall be $N_x=F_1(\Phi)=0, V=0, W=0,\ M_x=0$. .The function *W(x,y)* along the *y*-axis shall not change, therefore $\frac{\partial W}{\partial y}=0$ consequently, $\varepsilon_y=0$.

Thus, the conditions shall be fulfilled at $x=0,\ x=a$

$F_1(\Phi)=0,\ W=0,\ M_x=D(\chi_1+\mu\chi_2)=0,$

$\varepsilon_y=\frac{1}{Eh}(F_2(\Phi)-\mu F_1(\Phi))=0$.

Calculations

In the work (Petrov, 1975), stability calculation of the shallow shells of right-angular design (Lame parameters *A = 1, B = 1*) with pin-edge and movable fixing along the contour and being subjected to external uniformly distributed load is performed; moreover, the Bubnov–Galerkin method at one-term approximation of the required functions and the method of consecutive loadings (Petrov, 1975) are used for solution of equations of a mixed type. The critical load amounted to $\bar{P}_{kr}=509\left(\bar{P}=\frac{a^4q}{h^4E}\right)$ at values of the principal dimensionless curvatures of the shell $k_\xi=k_\eta=27\left(k_\xi=\frac{a^2}{hR}\right)$. The performed calculation of the same structure at constraint of 9 terms in expansion of the required functions gave the result $\bar{P}_{kr}=485$.

Further, the calculation of panels of steel toroid-shape shells (Lame parameters $A=r, B=d+r\sin x$) when using the fore-quoted equations of a mixed type was performed. The obtained values of critical loads for all the considered panels at *d* = 2 m are shown in Table 1.

Table 1. Critical loads for panels of steel toroid-shape shells

No.	*a*, rad	*b*, rad	*h*, m	*r*, m	q_{kr}, MPa
1	Pi/2	Pi/2	0.1	13	1.596
2	Pi/2	Pi/2	0.1	25	0.1448
3	Pi/2	Pi/2	0.1	5	3.7572
4	Pi/2	Pi/2	0.05	13	0.0308
5	Pi/2	Pi/2	0.05	25	0.3872
6	Pi/2	Pi/2	0.05	5	1.7608
7	Pi/2	Pi/2	0.025	13	0.3972
8	Pi/2	Pi/2	0.025	25	0.1936
9	Pi/2	Pi/2	0.025	5	0.8544

Thus, the critical load increases for panels of toroid-shape shells with increase in the shell thickness *h*, reduction in the radius of curvature *r* and increase in the angle of turn.

Conclusions

Shell constructions are used for covering of large span structures in the construction industry. Except for shallow shells of right-angular design, the panels of cylindrical, conical, toroidal shells and shallow spherical shells (domes) are widely used. The obtained equations for the shells of an arbitrary form, in which geometrical nonlinearity is taken into consideration, allow to easily select approximating functions at their calculation for pin-edge and movable fixing of the construction contour. Such a way of fixing of the shell contour allows to avoid stress concentration near edges of the construction, i. e. in the most dangerous area, where plastic and creep deformations develop most often.

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