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RITZ METHOD IN THE DISCRETE APPROXIMATION OF DISPLACEMENTS FOR SLAB CALCULATION

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Abstract
Introduction: The FEM reduces the problem of structural analysis for various building structures to the formation and solution of a system of linear algebraic equations. For this purpose, there are techniques available for obtaining FE stiffness and flexibility matrices where the main structural deformation characteristics are taken into account. However, the FEM can also be considered as a special case of the Ritz method in the discrete approximation of the required functions. In the functional of full potential deformation energy with regard to the considered structure, all adopted stress-strain state characteristics are taken into account.
Since it is difficult or impossible to find continuous approximation functions both in the classic version of the Ritz method and in the Bubnov–Galerkin method for some types of edge restraint in such building structures as beams, slabs, or shells, it is possible to use the Ritz method in the discrete approximation of the required functions (by analogy with the FEM). This paper presents a method of such calculations using slab calculations as an example. It is shown that, due to introducing some notations (operators), the process of finding the coefficients of the system of linear algebraic equations creates no difficulties and is easily programmable. The proposed method is not an alternative to the FEM, which is the most effective numerical method for the calculation of complex three-dimensional building structures.
Purpose of the study: We aimed to create a method for calculating slabs by the Ritz method in the discrete approximation of the deflection function for edge restraint cases when it is difficult or impossible to find continuous approximation functions in the classic version of the Ritz method and the Bubnov–Galerkin method. **Methods:** Based on the application of the Ritz variational method in the discrete approximation of displacements for slab calculation, all the basic relations for rectangular finite elements with 12 degrees of freedom are obtained, and an algorithm for forming the coefficients of the system of linear algebraic equations is developed. **Results:** For the first time, the solution by the Ritz method in the discrete approximation of slab displacements is obtained for the case when two edges of the slab are rigidly restrained and other two edges are free. In this case, the correct solution of the above problem is possible only with the use of the proposed method and FEM. For the test problem, we performed a comparison of the results of the calculation using the proposed method with the results using the classic Ritz method, which showed their very close agreement. The accuracy of the obtained results was assessed.

Keywords: Ritz method, functional of full potential deformation energy, discrete approximation of displacements, slab, deflection function, finite element, Hermite polynomials.

Introduction

In the early 1940s, the finite element method (FEM) was developed by utilizing the idea of the mesh method. This method originated from structural mechanics and the theory of elasticity, and was later comprehended by mathematicians who often call this method variational-difference, thus emphasizing its mathematical nature. Thanks to the works of Argyris (1961), Clough (1960), Courant (1943), Hrennikoff (1941), Zienkiewicz (1975) and others, this method has been widely used in calculations of various components of building constructions, buildings, and structures (Ilyin et al., 1990; Postnov and Kharkhurim, 1974; Trushin, 2018).

When slabs are calculated with the use of the FEM, generalized displacements q_i are introduced for each FE. Based on the type of potential deformation energy of the slab (plate), the stiffness matrix $\left[K\right]$ is found. If we introduce a vector of nodal displacements for FE $\left\{q\right\}$ = $\left\{q_1, q_2, \ldots, q_{12}\right\}$ and a vector of nodal forces $\left\{R\right\}$, based on the expression for work of external forces, then the relationship between these vectors according to Postnov et al. (1987) will have the following form: $\left\{R\right\} = \left[K\right]\left\{q\right\}$. In the work by Postnov et al. (1987), the expressions for the coefficients of the matrix $\left[K\right]$ are not given due to their cumbersomeness. In contrast to the FEM, the convergence of the solution by the Ritz method was proved (Mikhlin, 1970).

Since building structures are quite diverse and have different configurations and characteristics, different types of finite elements (FE) were developed (Auricchio et al., 2016, Bishay et al.,

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Farias et al., 2018, Zienkiewicz et al., 2013). Various works (Gander and Wanner, 2012; Li et al., 2014; Nwoji et al., 2017; Qu et al., 2013; Xue et al., 2021; Weinan and Yu, 2018) address the improvement of variational methods for the calculation of plates and shells and the development of modern computing systems based on them.

In construction practice, slabs having two opposite sides free are often used. When calculating such structures by the Ritz method in its classic version, it is impossible to find continuous approximation functions in this direction. Therefore, this paper proposes to use the Ritz method in the discrete approximation of displacement functions. The purpose of this study is to extend the scope of application of the Ritz method in solving new problems and develop a programming-friendly algorithm to calculate the coefficients of a system of linear algebraic equations.

Discrete approximation of the deflection function

Let us divide the area $D\{0 \le x \le a, 0 \le y \le b\}$ occupied by the middle plane of the slab into rectangular parts $D_{j,i}\{j = 1, 2, ..., m; i = 1, 2, ..., n\}$. Then we denote the points of intersection of these lines by $z_{j,i}$ (Fig. 1). Then we denote the area $D_{j,i}$ limited by points $z_{j,i}$, $z_{j,i+1}$, $z_{j+1,i}$, $z_{j+1,i+1}$ (nodal points) by $e_{j,i}$. Let us also denote the area D division interval in the direction of axis Ox by $h_x = \dfrac{a}{m}$ and in the direction of axis Oy — by $h_y = \dfrac{b}{n}$. The total number of the area D division points (nodal points $z_{j,i}$) will be $(m+1)(n+1) = mn + m + n + 1$, including internal nodal points ($(m-1)(n-1) = mn - m - n + 1$), and boundary nodal points ($2(m-1) + 2(n-1) + 4 = 2m + 2n$).

By analogy with the FEM, we will call the area $e_{j,i}$ a finite element (FE). The deflection function $W(x, y)$ and its partial derivatives $W'_x(x, y)$ and $W_y(x, y)$ will be considered unknown functions. To approximate these functions on the entire area D, we will first construct them on partial areas $D_{j,i}$, i.e., on FE $e_{j,i}$, ensuring continuity and differentiability of the obtained approximation of the required functions on the entire area D.

We will calculate the values of the required functions W, W'_x, W'_y at each nodal point $z_{j,i}$, considering those values unknown parameters. To approximate the required functions on FE e_{ji}, we will use third-degree splines (Ilyin et al., 1990) in the form of orthogonal Hermite polynomials (Korn and Korn, 1974). The most convenient form of such polynomials was described by Postnov and Kharkhurim (1974). From one-dimensional polynomials of variables *x* and *y, two-dimensional functions of variables x and y* φ_k^{ji}*(x, y) are formed*, and unknown functions $W(x, y)$, $W'_x(x, y), W'_y(x, y)$ on FE $e_{j,i}$ are presented as the sum of products of unknown numerical parameters (values of the required functions at nodal points) and known approximation functions φ_k^{ji}*(x, y).* On other FE, this approximation is considered to be equal to zero.

Since at each nodal point z_{ji} there will be three unknown numerical parameters, then FE e_{ji} will have 12 degrees of freedom.

The FE are connected to one another at the FE nodes. Let us necessitate the compatibility of vertical displacements $W(x, y)$ and rotation angles $W'_x(x, y), W'_y(x, y)$ at the nodal points for the FE adjacent to the node.

On FE $e_{j,i}$, let us denote the following:

$$
\begin{gathered}
W_{j,i} = W_1^{j,i},\ W_{j,i+1} = W_2^{j,i},\ W_{j+1,i} = W_3^{j,i},\\
W_{j+1,i+1} = W_4^{j,i},\ \left(W_{j,i}\right)'_x = W_5^{j,i},\\
\left(W_{j,i+1}\right)'_x = W_6^{j,i},\ \left(W_{j+1,i}\right)'_x = W_7^{j,i},\ \left(W_{j+1,i}\right)'_x = W_7^{j,i},\\
\left(W_{j+1,i+1}\right)'_x = W_8^{j,i},\\
\left(W_{j,i}\right)'_y = W_9^{j,i},\ \left(W_{j,i+1}\right)'_y = W_{10}^{j,i},\ \left(W_{j+1,i}\right)'_y = W_{11}^{j,i},\\
\left(W_{j+1,i+1}\right)'_y = W_{12}^{j,i}.
\end{gathered} \tag{1}
$$

Therefore, the deflection function $W(x, y)$ on FE $e_{j,i}$ can be represented as follows:

$$
W(x, y)|_{e_{j,i}} = \sum_{k=1}^{12} W_k^{j,i} \phi_k^{j,i}(x, y), \tag{2}
$$

and on other FE, this function is taken equal to zero.

On the entire area D, $W(x, y)$ is determined as $W_{m,n} = \sum_{j=0}^{m-1}\sum_{i=0}^{n-1}\sum_{k=1}^{12} W_k^{j,i}\phi_k^{j,i}(x, y)$. Each node will have three unknown parameters. In total, there will be $3(m+1)(n+1) = 3mn + 3m + 3n + 3$ unknown parameters.

To approximate the required functions, Hermite polynomials (Postnov and Kharkhurim, 1974) are used in the FEM. On FE $e_{j,i}$, they take the following form (in the common coordinate system xOy)

$$
\begin{gathered}
E_{01}^j(x) = \frac{h_x^3 - 3h_x\left(x - x_j\right)^2 + 2\left(x - x_j\right)^3}{h_x^3},\\
E_{02}^j(x) = \frac{3h_x\left(x - x_j\right)^2 - 2\left(x - x_j\right)^3}{h_x^3},\\
E_{11}^j(x) = \frac{h_x^2\left(x - x_j\right) - 2h_x\left(x - x_j\right)^2 + \left(x - x_j\right)^3}{h_x^2},\\
E_{12}^j(x) = \frac{-h_x\left(x - x_j\right)^2 + \left(x - x_j\right)^3}{h_x^2}.
\end{gathered} \tag{3}
$$

By substituting x with y, h_x with h_y, j with i, we can obtain $E_{01}^i(y)$, $E_{02}^i(y)$, $E_{11}^i(y)$, $E_{12}^i(y)$.

Let us introduce the following notations (Postnov and Kharkhurim, 1974):

$$
\phi_1^{j,i}(x, y) = E_{01}^j(x) \cdot E_{01}^i(y),
$$

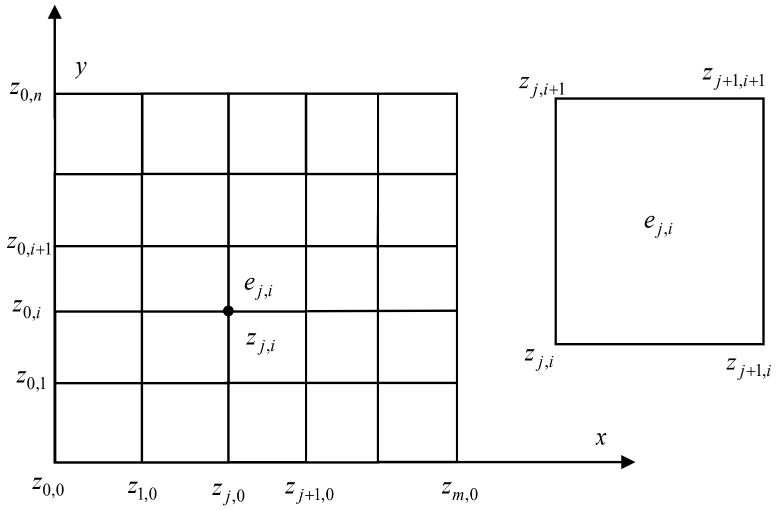

Fig. 1. Area D divided into FE

$$
\begin{aligned}
\phi_2^{j,i}(x,y) &= E_{01}^{j}(x)\cdot E_{02}^{i}(y),\\
\phi_3^{j,i}(x,y) &= E_{02}^{j}(x)\cdot E_{01}^{i}(y),\\
\phi_4^{j,i}(x,y) &= E_{02}^{j}(x)\cdot E_{02}^{i}(y),\\
\phi_5^{j,i}(x,y) &= E_{11}^{j}(x)\cdot E_{01}^{i}(y),\\
\phi_6^{j,i}(x,y) &= E_{11}^{j}(x)\cdot E_{02}^{i}(y),\\
\phi_7^{j,i}(x,y) &= E_{12}^{j}(x)\cdot E_{01}^{i}(y),\\
\phi_8^{j,i}(x,y) &= E_{12}^{j}(x)\cdot E_{02}^{i}(y),\\
\phi_9^{j,i}(x,y) &= E_{01}^{j}(x)\cdot E_{11}^{i}(y),\\
\phi_{10}^{j,i}(x,y) &= E_{01}^{j}(x)\cdot E_{12}^{i}(y),\\
\phi_{11}^{j,i}(x,y) &= E_{02}^{j}(x)\cdot E_{11}^{i}(y),\\
\phi_{12}^{j,i}(x,y) &= E_{02}^{j}(x)\cdot E_{12}^{i}(y).
\end{aligned}
\tag{4}
$$

The values of Hermite polynomials at the nodal points $z_{j,i}$, $z_{j,i+1}$, $z_{j+1,i}$, $z_{j+1,i+1}$ are 0 or 1. The values of the derivatives of $E_{01}^{j}(x)$ and $E_{02}^{j}(x)$ with respect to x and first-order derivatives of $E_{01}^{i}(y)$ and $E_{02}^{i}(y)$ with respect to y at nodal points are also equal to 0 or 1.

Each internal nodal point $z_{j,i}$ belongs to four FE (Fig. 2).

The order of numbering at the nodes of functions W, W_x', W_y' is shown in Fig. 2 by numbers. *Let us describe in detail the $W(x,y)$ approximation on each FE adjacent to the node $z_{j,i}$.* Below are the expressions $W(x,y)$ and approximating functions in formula (2) for each of the four FEs that have a common node $z_{j,i}$ (Fig. 2).

On FE $e_{j,i}$, in expansion (2), there will be expression $W(x,y)|_{e_{j,i}} = \sum_{k=1}^{12} W_k^{j,i}\phi_k^{j,i}(x,y)$, $W(z_{j,i})$ denoted by $W_1^{j,i}$ and, therefore, function $\phi_1^{j,i}(x,y)$, $W_x'(z_{j,i}) - W_5^{j,i}$ and $\phi_5^{j,i}(x,y)$, $W_y'(z_{j,i}) - W_9^{j,i}$ and $\phi_9^{j,i}(x,y)$, variation limits x and y will be $x_j \le x \le x_{j+1}$, $y_i \le y \le y_{i+1}$.

On FE $e_{j,i-1}$, in expansion (2), there will be expression $W(x,y)|_{e_{j,i-1}} = \sum_{k=1}^{12} W_k^{j,i-1}\phi_k^{j,i-1}(x,y)$, $W(z_{j,i})$ denoted by $W_2^{j,i-1}$ and, therefore, function $\phi_2^{j,i-1}(x,y)$, $W_x'(z_{j,i}) - W_6^{j,i-1}$ and $\phi_6^{j,i-1}(x,y)$, $W_y'(z_{j,i}) - W_{10}^{j,i-1}$ and $\phi_{10}^{j,i-1}(x,y)$, variation limits x and y will be $x_j \le x \le x_{j+1}$, $y_{i-1} \le y \le y_i$.

On FE $e_{j-1,i}$, in expansion (2), there will be expression $W(x,y)|_{e_{j-1,i}} = \sum_{k=1}^{12} W_k^{j-1,i}\phi_k^{j-1,i}(x,y)$, $W(z_{j,i})$ denoted by $W_3^{j-1,i}$ and, therefore, function $\phi_3^{j-1,i}(x,y)$, $W_x'(z_{j,i}) - W_7^{j-1,i}$ and $\phi_7^{j-1,i}(x,y)$, $W_y'(z_{j,i}) - W_{11}^{j-1,i}$ and $\phi_{11}^{j-1,i}(x,y)$, variation limits x and y will be $x_{j-1} \le x \le x_j$, $y_i \le y \le y_{i+1}$.

On FE $e_{j-1,i-1}$, in expansion (2), there will be expression $W(x,y)|_{e_{j-1,i-1}} = \sum_{k=1}^{12} W_k^{j-1,i-1}\phi_k^{j-1,i-1}(x,y)$, $W(z_{j,i})$ denoted by $W_4^{j-1,i-1}$ and, therefore, function $\phi_4^{j-1,i-1}(x,y)$, $W_x'(z_{j,i}) - W_8^{j-1,i-1}$ and $\phi_8^{j-1,i-1}(x,y)$, $W_y'(z_{j,i}) - W_{12}^{j-1,i-1}$ and $\phi_{12}^{j-1,i-1}(x,y)$, variation limits x and y will be $x_{j-1} \le x \le x_j$, $y_{i-1} \le y \le y_i$.

Method of obtaining algebraic equations

The functional of full potential deformation energy with regard to a rigid slab has the following form:

$$
E_s = \frac{D}{2}\int_0^a\int_0^b\left[\left(\frac{\partial^2 W}{\partial x^2}+\frac{\partial^2 W}{\partial y^2}\right)^2 + \right.
$$

$$
\left. +2(1-\mu)\left(\left(\frac{\partial^2 W}{\partial x\partial y}\right)^2 - \frac{\partial^2 W}{\partial x^2}\frac{\partial^2 W}{\partial y^2}\right) - 2\frac{q}{D}W\right]dxdy, \tag{5}
$$

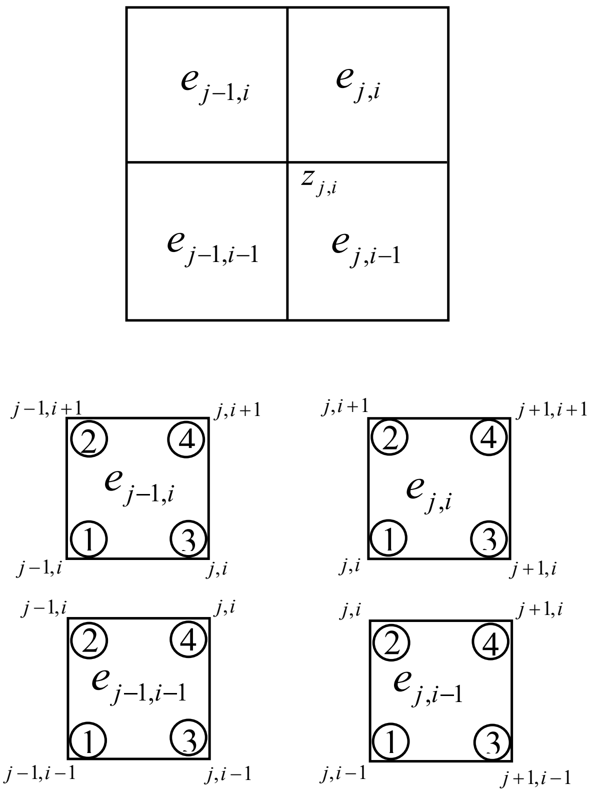

Fig. 2. FE adjacent to the node $z_{j,i}$

where $D=\dfrac{Eh^3}{12\left(1-\mu^2\right)}$ — cylindrical stiffness of the slab;

E — elasticity modulus;

h — slab height;

μ — Poisson's ratio;

W — slab deflection function;

q — transverse load;

a, b — slab dimensions in plan view.

The boundary conditions corresponding to the type of slab contour fixing are also specified. All this (the functional and boundary conditions) constitutes the variational problem to be solved.

On the entire area D occupied by the slab, the $W(x,y)$ approximation will have the following form:

$$W_{m,n}(x,y)=\sum_{j=0}^{m-1}\sum_{i=0}^{n-1}\sum_{k=1}^{12}W_k^{j,i}\phi_k^{j,i}(x,y). \tag{6}$$

To find the unknown parameters $W_k^{j,i}$, we substitute expression (6) into expression (5), and then find the derivatives of the functional $E_s\left(W_{m,n}(x,y)\right)$ with respect to the unknown parameters at each internal nodal point of the area D and equate them to 0. Each node will have three unknown parameters.

Thus, we obtain the following:

$$\frac{\partial E_s\left(W_{m,n}\right)}{\partial W\left(z_{j,i}\right)}=0,\ \frac{\partial E_s\left(W_{m,n}\right)}{\partial W'_x\left(z_{j,i}\right)}=0,\ \frac{\partial E_s\left(W_{m,n}\right)}{\partial W'_y\left(z_{j,i}\right)}=0, \tag{7}$$

when $j=1,2,...,m-1;\ i=1,2,...,n-1.$

Since there will be $(m-1)(n-1)$ internal nodal points, then there will be $3(m-1)(n-1)=3mn-3m-3n+3$ such equations.

The missing equations can be obtained using the boundary conditions.

Since each internal nodal point $z_{j,i}$ belongs to four FE at once, then each equation in (7) will contain four summands, according to the number of FE adjacent to the nodal point $z_{j,i}$.

Thus, the system of equations (7) can be written as follows:

$$\begin{gathered}
\frac{\partial E_s\left(e_{j,i}\right)}{\partial W_1{}^{j,i}}+\frac{\partial E_s\left(e_{j,i-1}\right)}{\partial W_2{}^{j,i-1}}+\frac{\partial E_s\left(e_{j-1,i}\right)}{\partial W_3{}^{j-1,i}}+\\
+\frac{\partial E_s\left(e_{j-1,i-1}\right)}{\partial W_4{}^{j-1,i-1}}=0,\\
\frac{\partial E_s\left(e_{j,i}\right)}{\partial W_5{}^{j,i}}+\frac{\partial E_s\left(e_{j,i-1}\right)}{\partial W_6{}^{j,i-1}}+\frac{\partial E_s\left(e_{j-1,i}\right)}{\partial W_7{}^{j-1,i}}+\\
+\frac{\partial E_s\left(e_{j-1,i-1}\right)}{\partial W_8{}^{j-1,i-1}}=0,\\
\frac{\partial E_s\left(e_{j,i}\right)}{\partial W_9{}^{j,i}}+\frac{\partial E_s\left(e_{j,i-1}\right)}{\partial W_{10}{}^{j,i-1}}+\frac{\partial E_s\left(e_{j-1,i}\right)}{\partial W_{11}{}^{j-1,i}}+\\
+\frac{\partial E_s\left(e_{j-1,i-1}\right)}{\partial W_{12}{}^{j-1,i-1}}=0.
\end{gathered} \tag{8}$$

In equations (8), the functional E_s is considered only on those FE that are adjacent to the internal node $z_{j,i}$ since the derivatives of the functional of other FE with respect to $W\left(z_{j,i}\right)$, $W'_x\left(z_{j,i}\right)$, $W'_y\left(z_{j,i}\right)$ will be 0.

Formation of the coefficients of the system of linear algebraic equations

The derivatives of functional (5) are taken with respect to the unknown parameters in each internal node of the area D.

At each nodal point, e.g., $z_{j,i}$, the three unknown parameters are the deflection value W, the value W'_x, and the value W'_y. Hence, the derivatives of the functional are taken with respect to the parameter W, parameter W'_x, parameter W'_y, and these derivatives are equated to 0 (Ritz method procedure). We obtain three equations at each internal node. There are four FE adjacent to the node $z_{j,i}$, each of which has the specified parameters in the common node $z_{j,i}$. Therefore, there will be four terms in each equation, which are derivatives of E_s with respect to the corresponding parameter included in each FE (see (8)).

The four FE adjacent to the node $z_{j,i}$ contain nine nodal points of the area ($z_{j,i}$, $z_{j+1,i}$, $z_{j,i+1}$, $z_{j,i-1}$, $z_{j-1,i}$, $z_{j-1,i-1}$, $z_{j-1,i+1}$, $z_{j+1,i+1}$, $z_{j+1,i-1}$). Therefore, in each of the three equations, there will be nine parameters of values W, nine parameters of values W'_x, and nine parameters of values W'_y, i.e., in each equation, there will be 27 terms and, accordingly, 27 coefficients, the values of which must be determined to solve the system of linear algebraic equations (SLAE), which is obtained after applying the described method to the initial variational problem.

For convenience of further transformations, we introduce the following notations (operators), based on the type of functional (5):

$$FW\left(f_1, f_2, \alpha, \beta\right) =$$
$$= \phi_{f_1xx}^{\alpha,\beta}\phi_{f_2xx}^{\alpha,\beta} + \phi_{f_1xx}^{\alpha,\beta}\phi_{f_2yy}^{\alpha,\beta} + \phi_{f_1yy}^{\alpha,\beta}\phi_{f_2xx}^{\alpha,\beta} + \phi_{f_1yy}^{\alpha,\beta}\phi_{f_2yy}^{\alpha,\beta} +$$
$$+2\left(1-\mu\right)\left(\phi_{f_1xy}^{\alpha,\beta}\phi_{f_2xy}^{\alpha,\beta} - \frac{1}{2}\phi_{f_1xx}^{\alpha,\beta}\phi_{f_2yy}^{\alpha,\beta} - \frac{1}{2}\phi_{f_1yy}^{\alpha,\beta}\phi_{f_2xx}^{\alpha,\beta}\right),$$

$$A_1^{j,i} = \int\limits_{x_j}^{x_{j+1}} dx \int\limits_{y_i}^{y_{i+1}} (\ )dy,\ A_2^{j,i} = \int\limits_{x_j}^{x_{j+1}} dx \int\limits_{y_{i-1}}^{y_i} (\ )dy,$$

$$A_3^{j,i} = \int\limits_{x_{j-1}}^{x_j} dx \int\limits_{y_i}^{y_{i-1}} (\ )dy,\ A_4^{j,i} = \int\limits_{x_{j-1}}^{x_j} dx \int\limits_{y_{i-1}}^{y_i} (\ )dy,$$

$$B_1^{j,i} = A_1^{j,i}\left(\phi_1^{j,i}\right) + A_2^{j,i}\left(\phi_2^{j,i-1}\right) +$$
$$+A_3^{j,i}\left(\phi_3^{j-1,i}\right) + A_4^{j,i}\left(\phi_4^{j-1,i-1}\right),$$
$$B_2^{j,i} = A_1^{j,i}\left(\phi_5^{j,i}\right) + A_2^{j,i}\left(\phi_6^{j,i-1}\right) +$$
$$+A_3^{j,i}\left(\phi_7^{j-1,i}\right) + A_4^{j,i}\left(\phi_8^{j-1,i-1}\right), \qquad (9)$$
$$B_3^{j,i} = A_1^{j,i}\left(\phi_9^{j,i}\right) + A_2^{j,i}\left(\phi_{10}^{j,i-1}\right) +$$
$$+A_3^{j,i}\left(\phi_{11}^{j-1,i}\right) + A_4^{j,i}\left(\phi_{12}^{j-1,i-1}\right).$$

Now all three equations of system (8) can be written as follows:

$$\sum_{k=1}^{12}\Big[A_1^{j,i}\left(W_k^{j,i}FW\left(k,1,j,i\right)\right) +$$
$$+A_2^{j,i}\left(W_k^{j,i-1}FW\left(k,2,j,i-1\right)\right) +$$
$$+A_3^{j,i}\left(W_k^{j-1,i}FW\left(k,3,j-1,i\right)\right) +$$
$$+A_4^{j,i}\left(W_k^{j-1,i-1}FW\left(k,4,j-1,i-1\right)\right)\Big] = \frac{q}{D}B_1^{j,i},$$

$$\sum_{k=1}^{12}\Big[A_1^{j,i}\left(W_k^{j,i}FW\left(k,5,j,i\right)\right) +$$
$$+A_2^{j,i}\left(W_k^{j,i-1}FW\left(k,6,j,i-1\right)\right) +$$
$$+A_3^{j,i}\left(W_k^{j-1,i}FW\left(k,7,j-1,i\right)\right) +$$
$$+A_4^{j,i}\left(W_k^{j-1,i-1}FW\left(k,8,j-1,i-1\right)\right)\Big] = \frac{q}{D}B_2^{j,i}, \quad (10)$$

$$\sum_{k=1}^{12}\Big[A_1^{j,i}\left(W_k^{j,i}FW\left(k,9,j,i\right)\right) +$$
$$+A_2^{j,i}\left(W_k^{j,i-1}FW\left(k,10,j,i-1\right)\right) +$$
$$+A_3^{j,i}\left(W_k^{j-1,i}FW\left(k,11,j-1,i\right)\right) +$$
$$+A_4^{j,i}\left(W_k^{j-1,i-1}FW\left(k,12,j-1,i-1\right)\right)\Big] = \frac{q}{D}B_3^{j,i}.$$

These equations ensure continuity of the approximation of functions $W\left(x,y\right)$ and $W_x'\left(x,y\right), W_y\left(x,y\right)$ on four FE adjacent to the node $z_{j,i}$.

We will denote the values of the parameters w_x' at the nodal points by wx, but it is necessary to add 4 to the indices of the parameter w, and, correspondingly, we will denote the values of the parameters w'_y at the nodal points by wy, but it is necessary to add 8 to the indices of w.

The first equation of system (10) can now be written in the following form:

$$a_1^{j,i}w_{j,i} + a_2^{j,i}w_{j+1,i} + a_3^{j,i}w_{j,i+1} + a_4^{j,i}w_{j,i-1} + a_5^{j,i}w_{j-1,i} +$$
$$+a_6^{j,i}w_{j-1,i-1} + a_7^{j,i}w_{j-1,i+1} + a_8^{j,i}w_{j+1,i+1} +$$
$$+a_9^{j,i}w_{j+1,i-1} + a_{10}^{j,i}wx_{j,i} + a_{11}^{j,i}wx_{j+1,i} + \ldots$$
$$+a_{18}^{j,i}wx_{j+1,i-1} + a_{19}^{j,i}wy_{j,i} + a_{20}^{j,i}wy_{j+1,i} + \ldots$$
$$+a_{27}^{j,i}wy_{j+1,i-1} = \frac{q}{D}B_1^{j,i}.$$

The first nine coefficients $a_1^{j,i} - a_9^{j,i}$ are basic and are as follows:

$$a_1^{j,i} = A_1^{j,i}\left(FW\left(1,1,j,i\right)\right) + A_2^{j,i}\left(FW\left(2,2,j,i-1\right)\right) +$$
$$+A_3^{j,i}\left(FW\left(3,3,j-1,i\right)\right) + A_4^{j,i}\left(FW\left(4,4,j-1,i-1\right)\right),$$
$$a_2^{j,i} = A_1^{j,i}\left(FW\left(3,1,j,i\right)\right) + A_2^{j,i}\left(FW\left(4,2,j,i-1\right)\right),$$
$$a_3^{j,i} = A_1^{j,i}\left(FW\left(2,1,j,i\right)\right) + A_3^{j,i}\left(FW\left(4,3,j-1,i\right)\right),$$
$$a_4^{j,i} = A_2^{j,i}\left(FW\left(1,2,j,i-1\right)\right) + A_4^{j,i}\left(FW\left(3,4,j-1,i-1\right)\right),$$
$$a_5^{j,i} = A_3^{j,i}\left(FW\left(1,3,j-1,i\right)\right) + A_4^{j,i}\left(FW\left(2,4,j-1,i-1\right)\right),$$
$$a_6^{j,i} = A_4^{j,i}\left(FW\left(1,4,j-1,i-1\right)\right),$$
$$a_7^{j,i} = A_3^{j,i}\left(FW\left(2,3,j-1,i\right)\right),$$
$$a_8^{j,i} = A_1^{j,i}\left(FW\left(4,1,j,i\right)\right),$$
$$a_9^{j,i} = A_2^{j,i}\left(FW\left(3,2,j,i-1\right)\right).$$

The parameter w has an index that changes from 1 to 4, and the value of f_1 in the operator $FW\left(f_1, f_2, \alpha, \beta\right)$ for that parameter changes as well. For the parameter wx, the index f_1 will change from 5 to 8. Hence, the coefficients $a_{10}^{j,i} - a_{18}^{j,i}$ are obtained from the corresponding coefficients $a_1^{j,i} - a_9^{j,i}$ by adding 4 to the corresponding value of f. Similarly, to obtain the coefficients $a_{19}^{j,i} - a_{27}^{j,i}$, we need to add 8 to the value f_1 in the corresponding coefficients $a_1^{j,i} - a_9^{j,i}$. In this case, in the first equation of system (10), the parameter f_2 in the operator $FW\left(f_1, f_2, \alpha, \beta\right)$ changes from 1 to 4. In the second equation of system (10), this parameter changes from 5 to 8, and in the third equation of system (10), f_2 changes from 9 to 12.

The second and third equations of system (10) can be briefly written in the following form:

$$b_1^{j,i} w_{j,i} + b_2^{j,i} w_{j+1,i} + b_3^{j,i} w_{j,i+1} + b_4^{j,i} w_{j,i-1} + b_5^{j,i} w_{j-1,i} +$$
$$+ b_6^{j,i} w_{j-1,i-1} + b_7^{j,i} w_{j-1,i+1} + b_8^{j,i} w_{j+1,i+1} + b_9^{j,i} w_{j+1,i-1} +$$
$$b_{10}^{j,i} wx_{j,i} + b_{11}^{j,i} wx_{j+1,i} + \ldots + b_{18}^{j,i} wx_{j+1,i-1} +$$
$$+ b_{19}^{j,i} wy_{j,i} + b_{20}^{j,i} wy_{j+1,i} + \ldots + b_{27}^{j,i} wy_{j+1,i-1} = \frac{q}{D} B_2^{j,i}.$$
$$c_1^{j,i} w_{j,i} + c_2^{j,i} w_{j+1,i} + \ldots + c_9^{j,i} w_{j+1,i-1} + c_{10}^{j,i} wx_{j,i} +$$
$$+ c_{11}^{j,i} wx_{j+1,i} + \ldots + c_{18}^{j,i} wx_{j+1,i-1} + c_{19}^{j,i} wy_{j,i} +$$
$$+ c_{20}^{j,i} wy_{j+1,i} + \ldots + c_{27}^{j,i} wy_{j+1,i-1} = \frac{q}{D} B_3^{j,i}.$$

To obtain the coefficients $b_1^{j,i} - b_{27}^{j,i}$, we need to add 4 to the value f_2 in the corresponding expressions of the coefficients $a_1^{j,i} - a_{27}^{j,i}$, and to obtain the coefficients $c_1^{j,i} - c_{27}^{j,i}$, we need to add 8 to the value f_2 in the corresponding expressions of the coefficients $a_1^{j,i} - a_{27}^{j,i}$. Thus, the algorithm for calculating the coefficients of the system of linear algebraic equations (SLAE) of the method for the discrete approximation of the initial functions can be presented in the form of Table, which will make it easy to design a program for their calculation with a computer.

When we moving to a new nodal point $z_{j,i}$, its coordinates x_j and y_i change. That is why these values should be changed in the expressions of the approximation of the functions $\varphi_k^{ji}(x, y)$ (4). Correspondingly, the coefficients of system (10) should be changed too. All these changes are carried out in a cycle with respect to variables *j, i* and do not pose any difficulties.

Calculation examples

As an example of the use of the considered method for slab calculation, let us find the deflection of a square slab with side a, which is under the uniformly distributed transverse load q. Let us assume that the slab has rigid restraint along the contour, therefore, on the contour, $w = 0,\ w'_x = 0,\ w'_y = 0$. The area $D\{0 \le x \le a; 0 \le y \le a\}$ is divided into four FE (Fig. 3). Due to the symmetry of the problem, at the node $z_{1,1}$, the first-order derivatives with respect to x and y will be equal to 0. Only the deflection at the node $z_{1,1}$ remains unknown.

In this case, the approximation $W(x,y)$ at each of the four FE will have the following form:

$$\left(h_x = \frac{a}{2},\ h_y = \frac{a}{2}\right)$$

On FE $e_{1,1}$: $W(x,y) = w_1^{1,1} \phi_1^{1,1}(x,y)$,

where $\frac{a}{2} \le x \le a,\ \frac{a}{2} \le y \le a,$

On FE $e_{1,0}$: $W(x,y) = w_2^{1,0} \phi_2^{1,0}(x,y)$,

where $\frac{a}{2} \le x \le a,\ 0 \le y \le \frac{a}{2},$

On FE $e_{0,1}$: $W(x,y) = w_3^{0,1} \phi_3^{0,1}(x,y)$,

where $0 \le x \le \frac{a}{2},\ \frac{a}{2} \le y \le a,$

On FE $e_{0,0}$: $W(x,y) = w_4^{0,0} \phi_4^{0,0}(x,y)$,

where $0 \le x \le \frac{a}{2},\ 0 \le y \le \frac{a}{2}.$

Here, $w_1^{1,1}$, $w_2^{1,0}$, $w_3^{1,0}$, $w_4^{0,0}$ are $w(z_{1,1})$.

The equation for finding $w(z_{1,1})$ will have the following form:

$$\frac{\partial f(e_{1,1})}{\partial w_1^{1,1}} + \frac{\partial f(e_{1,0})}{\partial w_2^{1,0}} + \frac{\partial f(e_{0,1})}{\partial w_3^{0,1}} + \frac{\partial f(e_{0,0})}{\partial w_4^{0,0}} = 0.$$

The compact form of this equation will be as follows (with the D multiplier omitted):

$$w(z_{1,1})\Big[A_1^0\big(FW(1,1,1,1)\big) + A_2^0\big(FW(2,2,1,0)\big) +$$
$$+ A_3^0\big(FW(3,3,0,1)\big) + A_4^0\big(FW(4,4,0,0)\big)\Big] = \frac{q}{D} B_1^0,$$

where

$$A_1^0(\ ) = \int_{a/2}^{a} dx \int_{a/2}^{a} ()dy,\ A_2^0(\ ) = \int_{a/2}^{a} dx \int_{0}^{a/2} ()dy,$$
$$A_3^0(\ ) = \int_{0}^{a/2} dx \int_{a/2}^{a} ()dy,\ A_4^0(\ ) = \int_{0}^{a/2} dx \int_{0}^{a/2} ()dy,$$

$$B_1^0 = A_1^0\left(\phi_1^{1,1}\right) + A_2^0\left(\phi_2^{1,0}\right) + A_3^0\left(\phi_3^{0,1}\right) + A_4^0\left(\phi_4^{0,0}\right).$$

Since there is one unknown parameter $w(z_{1,1})$ in the resulting equation, then the equation can be written in the following form:

Algorithm for calculating the SLAE coefficients for the slab

Coefficient No.	Coefficient type		
	$a_1^{j,i} - a_{27}^{j,i}$	$b_1^{j,i} - b_{27}^{j,i}$	$c_1^{j,i} - c_{27}^{j,i}$
1–9	Basic $1 \le f_1 \le 4; 1 \le f_2 \le 4$	$1 \le f_1 \le 4$ $f_2 = f_2 + 4$	$1 \le f_1 \le 4$ $f_2 = f_2 + 8$
10–18	$1 \le f_2 \le 4$ $f_1 = f_1 + 4$	$f_1 = f_1 + 4$ $f_2 = f_2 + 4$	$f_1 = f_1 + 4$ $f_2 = f_2 + 8$
19–27	$1 \le f_2 \le 4$ $f_1 = f_1 + 8$	$f_1 = f_1 + 8$ $f_2 = f_2 + 4$	$f_1 = f_1 + 8$ $f_2 = f_2 + 8$

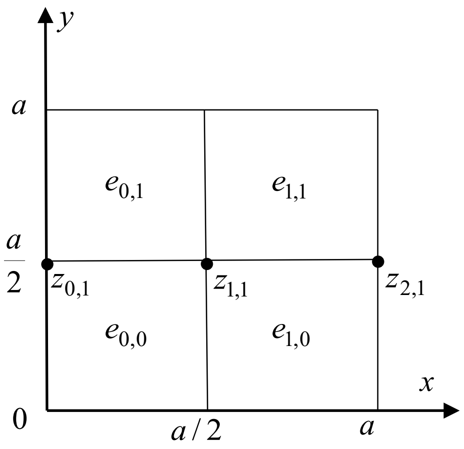

Fig. 3. Square slab divided into four FE

$$w\left(z_{1,1}\right)\cdot A=\frac{q}{D}B_1^0.$$

Let us calculate the integrals in the corresponding expressions.

Then we express the functions $\phi_1^{1,1}(x,y),\phi_2^{1,0}(x,y),\phi_3^{0,1}(x,y),\phi_4^{0,0}(x,y)$, using (4):

$$\phi_1^{1,1}=\frac{\left(\frac{a}{2}\right)^3-3\frac{a}{2}\left(x-\frac{a}{2}\right)^2+2\left(x-\frac{a}{2}\right)^3}{\left(\frac{a}{2}\right)^3}\times$$

$$\times\frac{\left(\frac{a}{2}\right)^3-3\frac{a}{2}\left(y-\frac{a}{2}\right)^2+2\left(y-\frac{a}{2}\right)^3}{\left(\frac{a}{2}\right)^3},$$

$$\phi_2^{1,0}=\frac{\left(\frac{a}{2}\right)^3-3\frac{a}{2}\left(x-\frac{a}{2}\right)^2+2\left(x-\frac{a}{2}\right)^3}{\left(\frac{a}{2}\right)^3}\times$$

$$\times\frac{3\frac{a}{2}y^2-2y^3}{\left(\frac{a}{2}\right)^3}, \qquad (11)$$

$$\phi_3^{0,1}=\frac{3\frac{a}{2}x^2-2x^3}{\left(\frac{a}{2}\right)^3}\cdot\frac{\left(\frac{a}{2}\right)^3-3\frac{a}{2}\left(y-\frac{a}{2}\right)^2+2\left(y-\frac{a}{2}\right)^3}{\left(\frac{a}{2}\right)^3},$$

$$\phi_4^{0,0}=\frac{3\frac{a}{2}x^2-2x^3}{\left(\frac{a}{2}\right)^3}\cdot\frac{3\frac{a}{2}y^2-2y^3}{\left(\frac{a}{2}\right)^3}.$$

The derivatives of the function $\phi_1^{1,1}(x,y)$ will take the following form:

$$\phi_{1xx}^{1,1}=\frac{-6\frac{a}{2}+12\left(x-\frac{a}{2}\right)}{\left(\frac{a}{2}\right)^3}\times$$

$$\times\frac{\left(\frac{a}{2}\right)^3-3\frac{a}{2}\left(y-\frac{a}{2}\right)^2+2\left(y-\frac{a}{2}\right)^3}{\left(\frac{a}{2}\right)^3},$$

$$\phi_{1yy}^{1,1}=\frac{\left(\frac{a}{2}\right)^3-3\frac{a}{2}\left(x-\frac{a}{2}\right)^2+2\left(x-\frac{a}{2}\right)^3}{\left(\frac{a}{2}\right)^3}\times$$

$$\times\frac{-6\frac{a}{2}+12\left(y-\frac{a}{2}\right)}{\left(\frac{a}{2}\right)^3},$$

$$\phi_{1xy}^{1,1}=\frac{-6\frac{a}{2}\left(x-\frac{a}{2}\right)+6\left(x-\frac{a}{2}\right)^2}{\left(\frac{a}{2}\right)^3}\times$$

$$\times\frac{-6\frac{a}{2}\left(y-\frac{a}{2}\right)+6\left(y-\frac{a}{2}\right)^2}{\left(\frac{a}{2}\right)^3}.$$

Then we calculate the integrals:

$$\int_{a/2}^{a}dx\int_{a/2}^{a}\Big[\left(\left(\phi_{1xx}^{1,1}\right)^2+2\phi_{1xx}^{1,1}\cdot\phi_{1yy}^{1,1}+\left(\phi_{1yy}^{1,1}\right)^2+\right.$$
$$\left.2(1-\mu)\left(\left(\phi_{1yy}^{1,1}\right)^2-\phi_{1xx}^{1,1}\cdot\phi_{1yy}^{1,1}\right)\Big]dy=\frac{47{,}177}{a^2},\right.$$

$$\int_{a/2}^{a}dx\int_{a/2}^{a}\phi_1^{1,1}(x,y)\,dy=\frac{a^2}{16}.$$

Other integrals are calculated in the same way. Thus, we obtain the following:

$$B_1^0=\frac{a^2}{4},\quad A=\frac{188{,}697}{a^2},$$

and, therefore, $w\left(z_{1,1}\right)=0{,}00132\dfrac{a^4q}{D}$.

For comparison, let us find the deflection of the slab under consideration using the Ritz method in the continuous approximation of $W(x,y)$ in the following form:

$$W(x,y)=w_1\sin^2\pi\frac{x}{a}\sin^2\pi\frac{y}{a}.$$

By substituting this expression into functional (5), we find the derivative of the functional with respect to w_1 and equate it to 0. As a result, we obtain the following equation:

$$\frac{\partial E_s}{\partial w_1}=D\int_0^a dx\int_0^a\Bigg[w_1\left(4\left(\frac{\pi}{a}\right)^4\cos^2 2\pi\frac{x}{a}\sin^4\pi\frac{y}{a}+\right.$$

$$+4\left(\frac{\pi}{a}\right)^4\sin^4\pi\frac{x}{a}\cos^2 2\pi\frac{y}{a}+$$

$$\left.+2\cdot 4\left(\frac{\pi}{a}\right)^4\cos 2\pi\frac{x}{a}\sin^2\pi\frac{y}{a}\sin^2\pi\frac{x}{a}\cos 2\pi\frac{y}{a}\right)+$$

$$+2(1-\mu)w_1\left(\left(\frac{\pi}{a}\right)^4\sin^2 2\pi\frac{x}{a}\sin^2 2\pi\frac{y}{a}-\right.$$

$$\left.-4\left(\frac{\pi}{a}\right)^4\cos 2\pi\frac{x}{a}\sin^2\pi\frac{x}{a}\sin^2\pi\frac{y}{a}\cos 2\pi\frac{y}{a}\right)-$$

$$-\frac{q}{D}\sin^2\pi\frac{x}{a}\sin^2\pi\frac{y}{a}\Bigg]dy = 0. \qquad (12)$$

Whence it follows that: $w_1 = w\left(z_{1,1}\right) = 0,00128\frac{qa^4}{D}$.

The exact solution of this problem is known:

$$W\left(\frac{a}{2},\frac{a}{2}\right) = 0,00126\frac{qa^4}{D}.$$

Thus, the solution obtained by the Ritz method in the continuous approximation of the deflection $W(x,y)$ differs from the exact one by 1.6%, and the solution obtained in the discrete approximation differs from the exact one by 4.6%.

The discrete approximation of the deflection is reasonable when the boundary conditions are such that it is difficult or impossible to find an approximation of the deflection by functions that are continuous over the entire area.

Now, in the example under consideration, let us change the slab edge restraint conditions Let us assume that at $y = 0$ and $y = a$, the edge is rigidly restrained, and at $x = 0, x = a$, it is free. Therefore, at the nodes $z_{1,1}, z_{0,1}, z_{2,1}$ (see Fig. 2), the deflection will not be 0. Moreover, at the nodes $z_{0,1}$ and $z_{2,1}$, w'_x will not be equal to 0, but w'_y at these nodes will be equal to 0 due to symmetry. Thus, $w\left(z_{1,1}\right), w\left(z_{0,1}\right), w\left(z_{2,1}\right), w'_x\left(z_{0,1}\right), w'_x\left(z_{2,1}\right)$ will be the sought parameters. Let us express the $W(x,y)$ approximation in this case for each of the four FE.

$$W(x,y)\Big|_{e_{1,1}} =$$

$$= w_1^{1,1}\phi_1^{1,1}(x,y) + w_3^{1,1}\phi_3^{1,1}(x,y) + w_7^{1,1}\phi_7^{1,1}(x,y),$$

$$W(x,y)\Big|_{e_{1,0}} =$$

$$= w_2^{1,0}\phi_2^{1,0}(x,y) + w_4^{1,0}\phi_4^{1,0}(x,y) + w_8^{1,0}\phi_8^{1,0}(x,y),$$

$$W(x,y)\Big|_{e_{0,1}} =$$

$$= w_3^{0,1}\phi_3^{0,1}(x,y) + w_1^{0,1}\phi_1^{0,1}(x,y) + w_5^{0,1}\phi_5^{0,1}(x,y),$$

$$W(x,y)\Big|_{e_{0,0}} =$$

$$= w_2^{0,0}\phi_2^{0,0}(x,y) + w_4^{0,0}\phi_4^{0,0}(x,y) + w_6^{0,0}\phi_6^{0,0}(x,y).$$

Here

$$w_1^{1,1}, w_2^{1,0}, w_3^{0,1}, w_4^{0,0} — w\left(z_{1,1}\right), w_3^{1,1}, w_4^{1,0} — w\left(z_{2,1}\right),$$

$$w_1^{0,1}, w_2^{0,0} — w\left(z_{0,1}\right),$$

$$w_7^{1,1}, w_8^{1,0} — w'_x\left(z_{2,1}\right), w_5^{0,1}, w_6^{0,0} — w'_x\left(z_{0,1}\right). 1$$

The functions $\phi_e^{\alpha,\beta}$ at α equal to 0 or 1 and β equal to 0 or 1 have form (4).

Let us denote the following:

$$w\left(z_{1,1}\right) = w_{1,1},\ w\left(z_{2,1}\right) = w_{2,1},\ w\left(z_{0,1}\right) = w_{0,1},$$

$$w'_x\left(z_{2,1}\right) = wx_{2,1},\ w'_x\left(z_{0,1}\right) = wx_{0,1}.$$

Now we can write the following:

$$W(x,y)\Big|_{e_{1,1}} = w_{1,1}\phi_1^{1,1} + w_{2,1}\phi_3^{1,1} + wx_{2,1}\phi_7^{1,1},$$

$$W(x,y)\Big|_{e_{1,0}} = w_{1,1}\phi_2^{1,0} + w_{2,1}\phi_4^{1,0} + wx_{2,1}\phi_8^{1,0},$$

$$W(x,y)\Big|_{e_{0,1}} = w_{1,1}\phi_3^{0,1} + w_{0,1}\phi_1^{0,1} + wx_{0,1}\phi_5^{0,1},$$

$$W(x,y)\Big|_{e_{0,0}} = w_{0,1}\phi_2^{0,0} + w_{1,1}\phi_4^{0,0} + wx_{0,1}\phi_6^{0,0}.$$

There will be one internal point $z_{1,1}$, therefore, there will be one equation:

$$\frac{\partial E_s\left(e_{1,1}\right)}{\partial w_{1,1}} + \frac{\partial E_s\left(e_{1,0}\right)}{\partial w_{1,1}} + \frac{\partial E_s\left(e_{0,1}\right)}{\partial w_{1,1}} + \frac{\partial E_s\left(e_{0,0}\right)}{\partial w_{1,1}} = 0.$$

By using the previously adopted notations (operators) and functional (5), we can write this equation as follows:

$$w_{1,1}A_1^0\left(FW(1,1,1,1)\right) + w_{2,1}A_1^0\left(FW(3,1,1,1)\right) +$$
$$+ wx_{2,1}A_1^0\left(FW(7,1,1,1)\right) + w_{1,1}A_2^0\left(FW(2,2,1,0)\right) +$$
$$+ w_{2,1}A_2^0\left(FW(4,2,1,0)\right) + wx_{2,1}A_2^0\left(FW(8,2,1,0)\right) +$$
$$+ w_{1,1}A_3^0\left(FW(3,3,0,1)\right) + w_{0,1}A_3^0\left(FW(1,3,0,1)\right) +$$
$$wx_{0,1}A_3^0\left(FW(5,3,0,1)\right) + w_{1,1}A_4^0\left(FW(4,4,0,0)\right) +$$
$$+ w_{0,1}A_4^0\left(FW(2,4,0,0)\right) + wx_{0,1}A_4^0\left(FW(6,4,0,0)\right) =$$
$$= \frac{q}{D}B_1^0.$$

Having calculated the corresponding integrals, we reduce this equation to the following form:

$$a_1 w_{1,1} + a_2 w_{2,1} + a_3 w_{0,1} + a_4 wx_{2,1} + a_5 wx_{0,1} = \frac{q}{D}a_b,$$

where

$$a_1 = 188,576\frac{1}{a^2},\quad a_2 = -46,35\frac{1}{a^2},\quad a_3 = -46,35\frac{1}{a^2},$$

$$a_4 = 16,811\frac{1}{a^2},\quad a_5 = -16,811\frac{1}{a^2},\quad a_6 = \frac{a^2}{4}.$$

Another four equations are obtained from the boundary conditions at the edge at $x = 0, x = a$.. Since these edges are free, the moment and transverse force must be 0 here, which means that at $x = 0, x = a$, the second-order derivative of the deflection w''_{xx} and the third-order derivative w'''_{xxx} must be 0. Thus, we obtain the following conditions:

$$w''_{xx}\left(z_{2,1}\right) = 0,\ w'''_{xxx}\left(z_{2,1}\right) = 0,\ w''_{xx}\left(z_{0,1}\right) = 0,$$

$$w'''_{xxx}\left(z_{0,1}\right) = 0.$$

And, therefore,

$$w''_{xx}\left(z_{2,1}\right)\Big|_{e_{1,1}} = 0,\ w'''_{xxx}\left(z_{2,1}\right)\Big|_{e_{1,1}} = 0,$$

$$w''_{xx}\left(z_{0,1}\right)\Big|_{e_{0,1}} = 0,\ w'''_{xxx}\left(z_{0,1}\right)\Big|_{e_{0,1}} = 0.$$

The missing four equations take the following form:

$$\frac{\partial^2 W}{\partial x^2}\Big|_{e_{1,1}} = w_1^{1,1}\frac{24}{a^2} - w_3^{1,1}\frac{24}{a^2} + w_7^{1,1}\frac{8}{a} = 0,$$

$$\frac{\partial^3 W}{\partial x^3}\Big|_{e_{1,1}} = w_1^{1,1}\frac{96}{a^3} - w_3^{1,1}\frac{96}{a^3} + w_7^{1,1}\frac{24}{a^2} = 0,$$

$$\frac{\partial^2 W}{\partial x^2}\Big|_{e_{0,1}} = -w_3^{0,1}\frac{24}{a^2} + w_1^{0,1}\frac{24}{a^2} - w_5^{0,1}\frac{8}{a} = 0,$$

$$\frac{\partial^3 W}{\partial x^3}\Big|_{e_{0,1}} = w_3^{0,1}\frac{96}{a^3} - w_1^{0,1}\frac{96}{a^3} + w_5^{0,1}\frac{24}{a^2} = 0.$$

Here

$$w_1^{1,1} = w_{1,1}, w_3^{1,1} = w_{2,1}, w_7^{1,1} = wx_{2,1},$$

$$w_3^{0,1} = w_{1,1}, w_1^{0,1} = w_{0,1}, w_5^{0,1} = wx_{0,1}.$$

The same equations are obtained if we use the following conditions:

$$w''_{xx}\left(z_{2,1}\right)\Big|_{e_{1,0}} = 0,\ w'''_{xxx}\left(z_{2,1}\right)\Big|_{e_{1,0}} = 0,\ w''_{xx}\left(z_{0,1}\right)\Big|_{e_{0,0}} = 0,$$

$$w'''_{xxx}\left(z_{0,1}\right)\Big|_{e_{0,0}} = 0.$$

Thus, to find the unknown parameters $w_{1,1}, w_{2,1}, w_{0,1}, wx_{2,1}, wx_{0,1}$, we have five equations:

$$-\frac{24}{a^2}w_{1,1} + \frac{24}{a^2}w_{0,1} - \frac{8}{a}wx_{0,1} = 0,$$

$$\frac{96}{a^3}w_{1,1} - \frac{96}{a^3}w_{0,1} + \frac{24}{a^2}wx_{0,1} = 0,$$

$$\frac{24}{a^2}w_{1,1} - \frac{24}{a^2}w_{2,1} + \frac{8}{a}wx_{2,1} = 0,$$

$$\frac{96}{a^3}w_{1,1} - \frac{96}{a^3}w_{2,1} + \frac{24}{a^2}wx_{2,1} = 0,$$

$$\frac{188,576}{a^2}w_{1,1} - \frac{46,35}{a^2}w_{2,1} - \frac{46,35}{a^2}w_{0,1} + \frac{16,811}{a^2}wx_{2,1} -$$

$$-\frac{16,811}{a^2}wx_{0,1} = \frac{a^2 q}{4D}.$$

If we add the first and third equations, we will obtain $wx_{0,1} = wx_{2,1}$, and if we subtract the fourth equation from the second equation, we will obtain $w_{0,1} = w_{2,1}$. The equality $w'_x\left(z_{0,1}\right) = w'_x\left(z_{2,1}\right)$ is possible only if these derivatives are 0. Therefore, since $w\left(z_{0,1}\right) = w\left(z_{2,1}\right)$, then $w\left(z_{1,1}\right) = w\left(z_{0,1}\right) = w\left(z_{2,1}\right)$. Given all this, based on the last equation, we obtain the following:

$$\frac{188,576}{a^2}w_{1,1} - \frac{96,7}{a^2}w_{1,1} = \frac{a^2 q}{4D},$$

therefore, $w_{1,1} = 0,0026\dfrac{a^4 q}{D}$.

By analyzing the obtained solution, we can conclude that the slab in this case deforms axisymmetrically, i.e., the deformation along the x axis is constant. In this case, the calculation for the deformation of the slab can be replaced by the calculation for the deformation of a beam of length a, rigidly fixed at the ends at $y = 0$, $y = a$. The equation of equilibrium of the beam will have the following form:

$$W^{\mathrm{IV}} = \frac{q}{EI},$$

and the general solution can be written as follows:

$$W\left(x\right) = \frac{q}{EI}\frac{y^4}{24} + c_1\frac{y^3}{6} + c_2\frac{y^2}{2} + c_3 y + c_4.$$

Based on the boundary conditions at $y = 0$, $y = a, W = 0, W' = 0$, we will obtain:

$$c_3 = 0,\ c_4 = 0,\ c_1 = -\frac{qa}{2EI},\ c_3 = \frac{qa^2}{12EI}.$$

Therefore, the deflection of the beam under consideration can be expressed by the following function:

$$W\left(x\right) = \frac{q}{EI}\left(\frac{y^4}{24} - \frac{a}{12}y^3 + \frac{a^2}{24}y^2\right),$$

and at $y = \dfrac{a}{2}$, the deflection will have the following form:

$$W\left(\frac{a}{2}\right) = 0,0026\frac{a^4 q}{EI}.$$

Conclusion

The system of linear algebraic equations obtained in the above examples contains 27 x 3 = 81 coefficients. And only the first nine coefficients are basic. Other coefficients can be found by recurrence relations based on these nine coefficients (see Table). Thus, due to the introduction of some notations (operators) *FW*, A_k^{ji}, B_k^{ji}, the process of finding the coefficients of the system of resolving algebraic equations is very simple and convenient for programming.

The proposed method of slab calculation by the Ritz method in the discrete approximation of displacements cannot serve as an alternative to the FEM, but it is very convenient for the calculation of relatively simple components of building structures, such as beams, slabs, and shells. However, the calculation of complex three-dimensional structures is possible only with the FEM.

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МЕТОД РИТЦА ПРИ ДИСКРЕТНОЙ АППРОКСИМАЦИИ ПЕРЕМЕЩЕНИЙ ДЛЯ РАСЧЕТА ПЛИТ

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Аннотация
Введение: МКЭ сводит задачу расчета самых различных строительных конструкций к формированию и решению системы линейных алгебраических уравнений. Для этого существуют методики получения матриц жесткости и податливости КЭ, в которых учитываются основные характеристики деформирования конструкции. Но МКЭ можно рассматривать и как частный случай метода Ритца при дискретной аппроксимации искомых функций. В функционале полной потенциальной энергии деформирования рассматриваемой конструкции учитываются все принятые характеристики напряженно-деформированного состояния. Так как для некоторых видов закрепления краев таких строительных конструкций как балка, плита или оболочка сложно или невозможно подобрать непрерывные аппроксимирующие функции как в классическом варианте метода Ритца, так и методе Бубнова – Галеркина, то (по аналогии с МКЭ) можно использовать метода Ритца при дискретной аппроксимации искомых функций. В работе на примере расчета плиты дается методика проведения таких расчетов. Показано, что введением некоторых обозначений-операторов процесс нахождения коэффициентов системы линейных алгебраических уравнений не вызывает затруднений и легко программируется. Предлагаемая методика не является альтернативой МКЭ, который является наиболее эффективным численным методом для расчета сложных трехмерных строительных конструкций. **Целью работы** было создание методики расчета плит методом Ритца при дискретной аппроксимации функции прогибов для случаев закрепления краев, когда сложно или невозможно подобрать непрерывные аппроксимирующие функции в классическом варианте метода Ритца и методе Бубнова – Галеркина. **Методы:** На основе применения вариационного метода Ритца при дискретной аппроксимации перемещений для расчета плит получены все основные соотношения для прямоугольных конечных элементов с 12 степенями свободы и разработан алгоритм формирования коэффициентов системы линейных алгебраических уравнений. **Результаты:** Впервые получено решение методом Ритца при дискретной аппроксимации перемещений плиты для случая, когда два края плиты жестко защемлены, а другие два края свободны. При этом корректное решение указанной задачи возможно только по предлагаемой методике и МКЭ. Для тестовой задачи было выполнено сравнение результатов расчета по предлагаемой методике с результатами при использовании классического метода Ритца, которое показало их весьма близкое совпадение. Оценена точность полученных результатов.

Ключевые слова: метод Ритца, функционал полной потенциальной энергии деформации, дискретная аппроксимация перемещений, плита, функция прогиба, конечный элемент, многочлены Эрмита.